Geometry/Topology Qualifying Exam Answer Bank

Zachary Slonim

Disclaimer: Hi all! The primary purpose of this document is to present a unified set of solutions to *all* of the questions on the last 17 years of UCLA Geometry/Topology qualifying exams. Almost every single answer represented here has been essentially copied directly from various other sources. As such, basically all of the credit goes to the following people: Ian Coley, Haris Khan, Jerry Luo, Gyu Eun Lee, Yan Tao, Thomas Martinez, Zach Baugher, Robert Miranda, Olha Shevchenko, Arian Nadjimzadah, John Hopper, Harahm Park, and William Chang. I have added a hint before each solution. This is what I personally think are the key aspects of the problem and what helped guide my review but should be taken with a grain of salt. I have also noted which other questions link to each question to give an idea of which sorts of problems are repeated the most and for easy navigation. Finally, I want to thank Karthik Sellakumaran Latha for proofreading.

Enjoy!

Fall 2024

This entire exam is an **exact replica** of Fall 2013.

Spring 2024

Spring 2024-1. (a) Show that the Lie group $SL_2(\mathbb{R}) = \{A \in M_{2\times 2}(\mathbb{R}) \mid \det(A) = 1\}$ is diffeomorphic to $S^1 \times \mathbb{R}^2$.

(b) Show that the Lie group $SL_2(\mathbb{C}) = \{A \in M_{2\times 2}(\mathbb{C}) \mid \det(A) = 1\}$ is diffeomorphic to $S^3 \times \mathbb{R}^3$.

(Hint to both parts: normalize the first row vector.)

Hint: Polar decomposition for both parts, orthogonal/unitary and positive definite hermitian/symmetric. Then $SO_2(\mathbb{R}) \cong S^1$ and $SU_2(\mathbb{C}) \cong S^3$.

This is exactly Fall 2012-1.

Spring 2024-2. Let

$$\mathbb{RP}^n = (\mathbb{R}^{n+1} - \{0\})/(x_0, \ldots, x_n) \sim t(x_0, \ldots, x_n),$$

for all $(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} - \{0\}$ and $t \in \mathbb{R} - \{0\}$ be the real n-dimensional projective space, and let $X = \{[(x_0, \ldots, x_n)] \in \mathbb{RP}^n \mid x_0 = 0\}$, where $[(x_0, \ldots, x_n)]$ is the equivalence class of $(x_0, \ldots, x_n)$. Is it possible to find a smooth map $f : \mathbb{RP}^n \to \mathbb{R}$ with $0 \in \mathbb{R}$ as a regular value and preimage $f^{-1}(0) = X$?

Hint: Not possible. $\mathbb{RP}^n - X$ is connected so f is either always positive or always negative on $\mathbb{RP}^n - X$ which means that it has local minima (or maxima) at points in X. Thus, $df = 0$ at points in X, contradicting 0 being a regular value and $f^{-1}(0) = X$.

No, this is not possible. Suppose $f : \mathbb{RP}^n \to \mathbb{R}$ was such a map. Note that we have

$$\mathbb{RP}^n - X = \{[(x_0, \ldots, x_n)] \in \mathbb{RP}^n \mid x_0 \neq 0\}.$$

This maps diffeomorphically to $\mathbb{R}^n$ by

$$[(x_0, \ldots, x_n)] \mapsto \left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right);$$

in fact, this map is exactly one of the natural charts that make $\mathbb{RP}^n$ into an n-manifold. Thus $\mathbb{RP}^n - X$ is connected. In particular, this implies that f is either always positive or always negative on $\mathbb{RP}^n - X$ since $\mathbb{R} - \{0\}$ has $\mathbb{R}^+$ and $\mathbb{R}^-$ for connected components and $f(\mathbb{RP}^n - X) \subset \mathbb{R} - \{0\}$ must be connected. Without loss of generality, assume f is always positive on $\mathbb{RP}^n - X$.

Now, we know that f is zero on X so f is always nonnegative, implying that f has (local) minima at all points in X. Take $p \in X$, let $v \in T_p\mathbb{RP}^n$ and let $\gamma : (-\epsilon, \epsilon) \to \mathbb{RP}^n$ be such that $\gamma(0) = p, \gamma'(0) = v$. Then,

$$df_p(v) = df_p(\gamma'(0)) = df_p(d\gamma_0(\frac{d}{dt})) = d(f \circ \gamma)_0(\frac{d}{dt}) = \frac{d}{dt}(f \circ \gamma)|_{t=0} = 0,$$

where the last equality comes from $f \circ \gamma$ having a (local) minimum at $t = 0$. But this was true for arbitrary $v \in T_p\mathbb{RP}^n$, showing that $df_p : T_p\mathbb{RP}^n \to T_0\mathbb{R} \cong \mathbb{R}$ is identically 0 so p is not a regular point of f, implying that 0 cannot be a regular value of f.

Spring 2024-3. Let $M \subset N$ a compact submanifold of codimension $\geqslant 3$. Show that if N is connected and simply connected, then so is the complement $N - M$.

Hint: Homotope a path in $\mathbb{R}^n$ to one that is transversal to M and show this must not intersect M. Use extension theorem on a homotopy from $[0,1] \times [0,1] \to M$ with $C = \{0,1\} \times [0,1] \cup [0,1] \times \{0,1\}$ a closed subset with the image of $H|_C$ transverse to M.

This is exactly Fall 2015-6 (changing $\mathbb{R}^n$ for N doesn't change anything as long as N is connected and simply connected).

Spring 2024-4. Let M, N be closed oriented n-manifolds with N connected. Show that if $f : M \to N$ has nonzero degree, then $f^* : H^*_{dR}(N;\mathbb{R}) \to H^*_{dR}(M;\mathbb{R})$ is injective. (Hint: First show that $f^* : H^n_{dR}(N;\mathbb{R}) \to H^n_{dR}(M;\mathbb{R})$ is injective.)

Hint: Use definition of degree as f^* being multiplication by $\deg(f)$ to show the hint. Then, use Poincaré duality, non-degenerate pairing.

First, we note that the degree of f satisfies

$$f^* : H^n_{dR}(N;\mathbb{R}) = \mathbb{R} \to H^n_{dR}(M;\mathbb{R}) = \mathbb{R} \text{ is given by } a \mapsto \deg(f) \cdot a.$$

I.e., on top cohomology, f^* is precisely multiplication by $\deg(f)$. Thus, if $\deg(f) \neq 0$, it is clear that $f^* : H^n_{dR}(N;\mathbb{R}) \to H^n_{dR}(M;\mathbb{R})$ is injective. Now, suppose that $\omega \in H^k_{dR}(N;\mathbb{R})$ is nonzero. Then, $\omega \wedge \eta \neq 0$ for some $\eta \in H^{n-k}_{dR}(N;\mathbb{R})$ by Poincaré duality as the pairing $H^k_{dR}(N;\mathbb{R}) \times H^{n-k}_{dR}(N;\mathbb{R}) \to H^n_{dR}(N;\mathbb{R})$ is non-degenerate. Since f^* is injective on top cohomology, we have $f^*(\omega) \wedge f^*(\eta) = f^*(\omega \wedge \eta) \neq 0$, implying that $f^*(\omega) \neq 0$ so indeed f^* is injective in all degrees of cohomology so is injective on the cohomology ring.

Spring 2024-5. Find two vector fields X and Y on $\mathbb{R}^3$ such that $X, Y, [X,Y]$ are everywhere linearly independent.

Hint: $X = e^x \frac{\partial}{\partial x}, Y = e^y \frac{\partial}{\partial y} + x, [X,Y] = 1$. Corresponding matrix is upper triangular with nowhere vanishing determinant.

Using the standard x, y, z coordinates on $\mathbb{R}^3$, take

$$X = e^x \frac{\partial}{\partial x}, \quad Y = e^y \frac{\partial}{\partial y} + x, \quad [X,Y] = 1.$$

At the point $p = (x,y,z) \in \mathbb{R}^3$, the matrix corresponding to $X_p, Y_p, [X,Y]_p$ is

$$A = \begin{pmatrix} e^x & x & 0 \\ 0 & e^y & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which has determinant $\det(A) = e^x e^y = e^{x+y}$. This is never 0 since $e^{x+y} > 0$ for all $x, y \in \mathbb{R}$ so these X_p, Y_p, and $[X,Y]_p$ are everywhere linearly independent.

Spring 2024-6. Let X be a topological space and $p \in X$. Let Y be the topological space obtained from $X \times [0,1]$ by contracting $(X \times \{0,1\}) \cup (\{p\} \times [0,1])$ to a point. Describe the relation between the homology groups of X and Y.

Hint: $\widetilde{H}_n(\Sigma X) = \widetilde{H}_n(SX) = \widetilde{H}_{n-1}(X)$ for all $n \geqslant 1$ and $H_0(\Sigma X) = \mathbb{Z}$. Because $\Sigma X = SX/(\{p\} \times [0,1])$ so look at long exact sequence. Get $\widetilde{H}_n(SX)$ from Mayer-Vietoris.

This is exactly Spring 2016-9.

Spring 2024-7. Exhibit a space whose fundamental group is isomorphic to $(\mathbb{Z}/m\mathbb{Z}) * (\mathbb{Z}/n\mathbb{Z})$, where $\mathbb{Z}/k\mathbb{Z}$ denotes the integers modulo k and $*$ denotes the free product. Also exhibit a space whose fundamental group is isomorphic to $(\mathbb{Z}/m\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z})$.

Hint: Attach via $z \mapsto z^k$ to get $\mathbb{Z}/k\mathbb{Z}$. Wedge sum and product of these spaces.

This is exactly Spring 2016-7.

Spring 2024-8. (a) Define what it means for a covering space to be regular.
(b) Give an example of an irregular covering space of the wedge sum $S^1 \vee S^1$.

Hint: Subgroup $\langle a^2, b^2, aba^{-1}, bab^{-1}\rangle \subset \langle a, b\rangle$. Corresponding cover has three vertices with a loop on left one, b loop on right one and base point in the middle.

This is exactly Spring 2014-7.

Spring 2024-9. (a) Show that a nonsingular linear $A : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ induces a smooth map $\Phi_A : \mathbb{CP}^n \to \mathbb{CP}^n$.
(b) Show that the fixed point of Φ_A correspond to eigenvectors of the original matrix.
(c) Show that Φ_A is a Lefschetz map if the eigenvalues of A all have multiplicity 1.
(d) Show that the Lefschetz number of Φ_A is $n + 1$.

Hint: $(d\phi_A)_x$ has no fixed points for any eigenvector x of A. Use local coordinates. Since we can homotope anywhere by connectedness, use $A = \text{diag}(1, 2, \dots, n+1)$ to use part (c) so Lefschetz number is sum of local Lefschetz numbers which are all $+1$ so answer is $n + 1$.

This is exactly Spring 2013-6.

Spring 2024-10. Consider the following subsets of $\mathbb{R}^3$:

$$A = \{(0, 0, z) \mid z \in \mathbb{R}\},$$

$$B = \{(\cos\theta, \sin\theta, 0) \mid \theta \in \mathbb{R}\},$$

$$C = \{(\cos\theta, \sin\theta + 5, 0) \mid \theta \in \mathbb{R}\}.$$

Show that $\mathbb{R}^3 - A - B$ and $\mathbb{R}^3 - A - C$ are not homeomorphic.

Hint: Fundamental groups of $\mathbb{R}^3 - (A \cup B)$ and $\mathbb{R}^3 - (A \cup C)$ are different.

This is essentially the same as Fall 2021-10.

Fall 2023

Fall 2023-1. Consider the space of all straight lines in $\mathbb{R}^2$ (not necessarily those passing through the origin). Explain how to give it the structure of a smooth manifold. Is it orientable?

Hint: $f : X \to \mathbb{RP}^2$, $ax + by + c = 0 \mapsto [a : b : c] \in \mathbb{RP}^2$ is a bijection onto its image which is $\mathbb{RP}^2 - \{[0 : 0 : 1]\}$. Not orientable since $\mathbb{RP}^2$ is not, double cover $S^2 \to \mathbb{RP}^2$.

This is exactly Spring 2016-1.

Fall 2023-2. Let ω be a closed 2-form on a smooth manifold M and let X, Y be smooth vector fields on M. Show that if $i_X\omega = i_Y\omega = 0$, then $i_{[X,Y]}\omega = 0$.

Hint: Show $i_{[X,Y]}\omega = [L_X, i_Y]$: Compute right hand side using $(i_Y\omega)(V_1, \ldots, V_{k-1}) = \omega(Y, V_1, \ldots, V_{k-1})$ and $(L_X\omega)(V_1, \ldots, V_k) = X(\omega(V_1, \ldots, V_k)) - \sum_{i=1}^k \omega(V_1, \ldots, V_{i-1}, [X, V_i], V_{i+1}, \ldots, V_k)$. Or use chain rule for Lie derivative. Then Cartan's magic formula and assumptions finish it.

By Spring 2020-4, we have

$$i_{[X,Y]}\omega = [L_X, i_Y]\omega = L_X i_Y\omega - i_Y L_X\omega = -i_Y(di_X\omega + i_X d\omega) = 0,$$

where we have used Cartan's magic formula, the fact that $d\omega = 0$ since ω is closed, and the assumption that $i_X\omega = i_Y\omega = 0$.

Fall 2023-3. Consider the map $d_f : \Omega^i(M) \to \Omega^{i+1}(M)$ given by $\omega \mapsto d\omega + df \wedge \omega$, where M is a smooth manifold, $\Omega^i(M)$ is the set of smooth i-forms on M, and f is a smooth function on M.

(a) Show that d_f is a cochain map, i.e., $d_f \circ d_f = 0$.
(b) Let $H^i_f(M)$ be the ith cohomology group of the cochain complex $(\Omega^i(M), d_f)$. Show that $H^0_f(M) \cong \mathbb{R}$ when M is the real line $\mathbb{R}$.

Hint: Just directly compute. $\Omega^1(\mathbb{R}) = \{gdx \mid g \in \Omega^0(\mathbb{R})\}$. Then $d : \Omega^0(\mathbb{R}) \to \Omega^1(\mathbb{R})$ is just taking the derivative under this identification so $d_f : g \mapsto g' + gf'$ has kernel $\{ae^{-f} \mid a \in \mathbb{R}\}$ which is isomorphic to $\mathbb{R}$.

(a) Let $\omega \in \Omega^i(M)$. We compute

$$\begin{aligned} d_f(d_f(\omega)) = d_f(d\omega + df \wedge \omega) &= d(d\omega + df \wedge \omega) + df \wedge (d\omega + df \wedge \omega) \\ &= dd\omega + ddf \wedge \omega - df \wedge d\omega + df \wedge d\omega + df \wedge df \wedge \omega \\ &= 0 + 0 - df \wedge d\omega + df \wedge d\omega + 0 = 0, \end{aligned}$$

where we use the facts that $d^2 = 0$ and that $df \wedge df = 0$ since df is a 1-form.

(b) By definition, $\Omega^{-1}(M) = 0$ so

$$H^0_f(M) = \ker(d_f : \Omega^0(M) \to \Omega^1(M)).$$

With $M = \mathbb{R}$, we know that $\Omega^0(\mathbb{R}) = C^\infty(\mathbb{R})$ is simply the group of smooth functions from $\mathbb{R}$ to $\mathbb{R}$ under addition. By Poincaré duality, we can canonically identify $\Omega^1(\mathbb{R})$ with the group of smooth functions from $\mathbb{R}$ to $\mathbb{R}$ since every 1-form on $\mathbb{R}$ is equal to fdx for a unique smooth function $f : \mathbb{R} \to \mathbb{R}$.

With this identification, we note that the standard differential operator $d : \Omega^0(\mathbb{R}) \to \Omega^1(\mathbb{R})$ sends a function g to its standard derivative g'. So d_f sends g to $g' + gf'$. We can solve $g' + gf' = 0$ using integrating factor to get $\ker(d_f) = \{ae^{-f} \mid a \in \mathbb{R}\}$. Clearly this is isomorphic to $\mathbb{R}$ via $a \mapsto ae^{-f}$.

Fall 2023-4. Let X and Y be submanifolds of $\mathbb{R}^n$. Prove that, for almost all $a \in \mathbb{R}^n$, the translate $X + a := \{x + a \mid x \in X\}$ intersects Y transversely.

Hint: Show $F : X \times \mathbb{R}^n \to \mathbb{R}^n, F(x, a) = x + a$ is transverse to Y. Thom's transversality theorem.

This is exactly Spring 2016-2.

Fall 2023-5. Let $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the 2-dimensional torus and let C be the curve which is the image of the line $\{2x - 5y = 0\} \subset \mathbb{R}^2$ under the projection $\mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$.

(a) Write a differential form on T^2 which represents the Poincaré dual to C.

(b) Is there a differential form which represents the Poincaré dual to C and is zero on a neighborhood of the point $(0, 0) \in T^2$?

Hint: Expand and solve in terms of the generators of the exterior algebra dx and dy. (b): Ball diffeomorphic to to open ball in $\mathbb{R}^n$. Bump function ϕ. $i^*(\omega)$ closed implies exact so find η with $d\eta = i^*(\omega)$. $\tau = \omega - d(\phi\eta)$ works.

(a) This is exactly the same as Spring 2014-5 but with the line $2x = 5y$ instead so the dual is $2dx - 5dy$.

(b) Yes: since T^2 is a closed manifold, this follows from Spring 2022-2.

Fall 2023-6. Compute the integral homology groups of the complex projective space $\mathbb{CP}^n$. If n is even, prove that it does not cover any manifold except itself.

Hint: Attach by $\phi_n : e^{2n} \to \mathbb{CP}^n$, $\quad (z_0, \dots, z_{n-1}) \mapsto [z_0, \dots, z_{n-1}, t]$, $\quad t = \sqrt{1 - \sum_{i=1}^{n-1} z_i\overline{z_i}}$. $H_*(\mathbb{CP}^n) = \mathbb{Z}_{(0)} \oplus \mathbb{Z}_{(2)} \oplus \cdots \oplus \mathbb{Z}_{(2n)}$. $\pi_1(X)$ acts on $\mathbb{CP}^{2n}$ via deck transformations. Any map $g : \mathbb{CP}^{2n} \to \mathbb{CP}^{2n}$ has a fixed point by Lefschetz trace formula so they are all the identity.

(a) This was done in Spring 2021-5.

(b) This is exactly Spring 2020-5.

Fall 2023-7. Let $X = \Sigma_g$ and $Y = \Sigma_h$ be surfaces of genus g and h respectively, with $0 < g < h$. Prove that every map $X \to Y$ induces the zero map on the second homology H_2. Construct a map $X \to Y$ which induces a non-zero map on the first homology H_1.

Hint: Suffices to show $H^2(Y) \to H^2(X)$ is zero by universal coefficient theorem. Use the cohomology ring structure on genus g surface which has $2g$ generators in degree 1 so that all pairwise products are 0 except $a_ib_i = \sigma$, the generator in degree 2. Then $h > g$ forces the induced map to be zero. Map $X \to \Sigma^1 \to S^1 \to Y$ in such a way that a loop in X is sent to a nontrivial loop in Y.

We know the homology of a surface of genus g is

$$H_k(\Sigma_g) = \begin{cases} \mathbb{Z} & k = 0, 2 \\ \mathbb{Z}^{2g} & k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Since these are all free, it suffices to show that the induced map $H^2(Y) \to H^2(X)$ is zero as then so too is the map $H_2(X) \to H_2(Y)$ by the universal coefficient theorem. For this, we note that the cohomology ring structure of Σ_g is generated as a $\mathbb{Z}$-module by the element 1 in degree 0, the elements $a_1, \dots, a_g, b_1, \dots b_g$ in degree 1, and the element σ in degree 2. This then has product structure $a_ia_j = b_ib_j = 0$ for all i, j and $a_ib_j = -b_ja_i = \delta_{ij}\sigma$ where δ_{ij} is the Kronecker delta function.

Now, note that $\dim(H^1(Y)) = 2h > 2g = \dim(H^1(X))$ so $H^1(Y) \to H^1(X)$ has nonzero kernel. Let $x = c_1a_1 + \dots + c_ha_h + d_1b_1 + \dots + d_hb_h$ be in the kernel and without loss of generality, suppose $c_1 \neq 0$. Then, $xb_1 = c_1\sigma \neq 0$ is a generator of $H^2(Y)$. But $c_1f^*(\sigma) = f^*(c_1\sigma) = f^*(xb_1) = f^*(x)f^*(b_1) = 0$ so $f^* : H^2(Y) \to H^2(X)$ must be the zero map.

For the second part, we first map $X \to S^1$ in a way that induces a nonzero map on first homology. This is done by composing a surjective continuous map $X \to \Sigma_1$ which is nonzero on first homology with the classic surjection $\Sigma_1 \to S^1$ that is projection onto the circle that goes around the inside of the donut. Finally, we map $S^1 \to Y$ making sure that we hit a loop that is a generator of $H_1(Y)$ such that this induced composition map is nonzero on first homology.

Fall 2023-8. Consider the following group with $2n$ generators and 1 relation

$$G_n = \langle a_1, b_1, a_2, b_2, \dots, a_n, b_n \mid a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1} \cdots a_nb_na_n^{-1}b_n^{-1} \rangle.$$

For which pairs (m, n) does G_n contain a finite index subgroup isomorphic to G_m?

Hint: $m = 1 + k(n-1)$ for some $k \in \mathbb{N}$. Corresponds to coverings of an n-torus, which are exactly the m-tori such that $\chi(T_n) = 2 - 2n \mid 2 - 2m = \chi(T_m)$.

This is exactly Spring 2020-9.

> **Fall 2023-9.** Define the orientation double cover of a manifold. Explicitly identify the space which is the orientation double cover of the real projective plane $\mathbb{RP}^n$. (Hint: $\mathbb{RP}^n$ is the quotient of S^n by the antipodal map; is the antipodal map orientation-preserving or orientation-reversing?)

Hint: Unique two-fold orientable covering space with orientation reversing non-trivial deck transformation. $\mathbb{RP}^n \sqcup \mathbb{RP}^n$ with interchanging copies if n is odd and S^n with antipodal map if n is even.

This is exactly Spring 2021-6.

> **Fall 2023-10.** Let D^2 be the unit disk in $\mathbb{C}$, and let $S^1 = \partial D^2$. let $X = D^2 \times S^1 \times \{0,1\}/\sim$ where
>
> $$(x, y, 0) \sim (xy^5, y, 1)$$
>
> for all $x, y \in S^1$. Compute the homology groups of X.

Hint: $H_*(X) = \mathbb{Z}_{(3)} \oplus \mathbb{Z}_{(2)} \oplus \mathbb{Z}_{(1)} \oplus \mathbb{Z}_{(0)}$. Mayer Vietoris for neighborhoods around $D^2 \times S^1 \times \{0\}, D^2 \times S^1 \times \{1\}$. Then $A \cap B \simeq \partial D^2 \times S^1 = S^1 \times S^1$.

This is exactly Spring 2020-10.

Spring 2023

> **Spring 2023-1.** Let M, N be smooth manifolds and $F : M \to N$ a smooth proper map.
>
> (a) Show that F maps closed sets to closed sets.
> (b) Show that the set of regular values is open.
> (c) Let $C \subset N$ be compact. Show that for every open set $U \subset M$ containing $F^{-1}(C)$ there is an open set $V \subset N$ containing C, such that $F^{-1}(V) \subset U$.

Hint: Sequence definition of closed. Show regular points is open, used closedness. Let $V = (F(U^c))^c$.

(a) By definition of being proper, the inverse image of a compact set $V \subset N$ is compact in M. Let $A \subset M$ be closed. Let y_n be a sequence of points in $F(A)$ such that $y_n \to y$ (where we inherently use the metrizability of any smooth manifold). As $y_n \in F(A)$, there is some $x_n \in A$ for each n such that $F(x_n) = y_n$. The set $K = \{y_n\}_{n \in \mathbb{N}} \cup \{y\}$ is compact in N since any open cover contains a neighborhood of y which contains infinitely many of the points and then we can just take finitely many opens to cover the rest of the points.

Hence $F^{-1}(K)$ is compact and moreover each $x_n \in F^{-1}(K)$. Now, $F^{-1}(K)$ is sequentially compact as in a metrizable space, compact and sequentially compact are equivalent, so there is some $x \in F^{-1}(K)$ and a subsequence of x_n so that $x_{n_k} \to x$. Since F is continuous, we know that $F(x_{n_k}) \to F(x)$ but $F(x_{n_k}) = y_{n_k} \to y$ so $F(x) = y$ by uniqueness of limits in Hausdorff spaces. Hence, $y \in F(A)$ so $F(A)$ is closed as it contains all of its limit points.

(b) Let $A \subset M$ be the set of regular points of F and $B \subset N$ be the set of regular values. Let $x \in A$ and choose coordinate charts ϕ, ψ centered at $x, F(x)$ respectively. Since x is a regular point, the derivative $d(\psi \circ F \circ \phi^{-1})$ is surjective at $\phi(x)$. So there is an $n \times n$ minor of the matrix with non-zero determinant (where $n = \dim(N)$). But det is a continuous map so there is a neighborhood of $\phi(x)$ so that $d(\psi \circ f \circ \phi^{-1})$ is surjective for every point in the neighborhood (as the same $n \times n$ minor has non-zero determinant). Since ϕ is a diffeomorphism, this means there is a neighborhood of x for which F has rank n at each point in the neighborhood. This neighborhood is a subset of A so A is open.

Now, note that $y \in B$ if and only if $F^{-1}(y) \subset A$ by definition of being a regular value. So $y \in B$ if and only if $F(x) \neq y$ for all $x \in A^c$ which is equivalent to $y \notin F(A^c)$ so $B = (F(A^c))^c$. A is open so A^c is closed so $F(A^c)$ is closed since F is a closed map, so B is open as desired.

(c) Let $V = (F(U^c))^c$ which is open by the same logic as in the last paragraph. Suppose that $C \not\subset V$. Then, there is some $y \in C$ such that $y \notin V$. So $y \in F(U^c)$ implying that there is some $x \in U^c$ with $F(x) = y$. But $y \in C$ so $x \in F^{-1}(C) \subset U$, a contradiction. Hence, $C \subset V$. Similarly, suppose that $F^{-1}(V) \not\subset U$. Then, there is some $x \in F^{-1}(V)$ such that $x \notin U$. So $F(x) \in V$ means that $F(x) \notin F(U^c)$, contradicting the fact that $x \notin U$. Hence, $F^{-1}(V) \subset U$, showing this V is as desired.

Spring 2023-2. Consider a smooth map $F : \mathbb{CP}^n \to \mathbb{CP}^n$.

(a) When n is even show that F has a fixed point.
(b) When n is odd give an example where F does not have a fixed point.

Hint: Lefschetz fixed point theorem. $H^*(\mathbb{CP}^{2k};\mathbb{Z}) = \frac{\mathbb{Z}[\alpha]}{\alpha^{2k+1}}$, $|\alpha| = 2$ so $L(F) = \sum_{j=0}^{2k} m^j$ for some $m \in \mathbb{Z}$. Counterexample is $[z_0 : z_1 : \ldots : z_n : z_{n+1}] \mapsto [-\overline{z_1} : \overline{z_0} : \ldots : -\overline{z_{n+1}} : \overline{z_n}]$.

Referenced in: Spring 2020-5, Spring 2011-9.

(a) By the Lefschetz fixed point theorem, it suffices to show that $L(F) \neq 0$. Let $n = 2k$. We know that the cohomology ring for $\mathbb{CP}^{2k}$ is

$$H^*(\mathbb{CP}^{2k};\mathbb{Z}) = \frac{\mathbb{Z}[\alpha]}{\alpha^{2k+1}}, \quad |\alpha| = 2.$$

Note that we only have nonzero cohomology in even degree. Let $m \in \mathbb{Z}$ be such that $F^*(\alpha) = m\alpha$. By the cup product (multiplicative structure) of the cohomology ring, we have

$$F^*(\alpha^j) = m^j \alpha^j,$$

where α^j is a generator of $H^{2j}(\mathbb{CP}^{2k};\mathbb{Z})$. Moreover, by the above calculation, we know that

$$\mathrm{Tr}(F^* : H^{2j}(\mathbb{CP}^{2k};\mathbb{Q}) \to H^{2j}(\mathbb{CP}^{2k};\mathbb{Q})) = \mathrm{Tr}(F^* : H^{2}(\mathbb{CP}^{2k};\mathbb{Q}) \to H^{2}(\mathbb{CP}^{2k};\mathbb{Q}))^j = m^j.$$

Thus, we have $L(F) =$

$$\sum_{j=0}^{4k} (-1)^j \mathrm{Tr}(F^* : H^{j}(\mathbb{CP}^{2k};\mathbb{Q}) \to H^{j}(\mathbb{CP}^{2k};\mathbb{Q})) = \sum_{j=0}^{2k} \mathrm{Tr}(F^* : H^{2j}(\mathbb{CP}^{2k};\mathbb{Q}) \to H^{2j}(\mathbb{CP}^{2k};\mathbb{Q})) = \sum_{j=0}^{2k} m^j.$$

If $m = 1$, then clearly $L(F) \neq 0$. If $m \neq 1$, then

$$\sum_{j=0}^{2k} m^j = \frac{m^{2k+1} - 1}{m - 1} \neq 0$$

since $2k + 1$ is odd.

(b) Write $n = 2k - 1$ and let $F : \mathbb{C}^{2k} - \{0\} \to \mathbb{C}^{2k} - \{0\}$ be defined by

$$(z_0, z_1, \ldots, z_{2k-2}, z_{2k-1}) \mapsto (-\overline{z_1}, \overline{z_0}, \ldots, -\overline{z_{2k-1}}, \overline{z_{2k-2}}).$$

Then $F(\lambda z) = \overline{\lambda} F(z)$ so F factors through the projection map $\pi : \mathbb{C}^{2k} - \{0\} \to \mathbb{CP}^{2k-1}$ to give a map $\widetilde{F} : \mathbb{CP}^{2k-1} \to \mathbb{CP}^{2k-1}$ defined by

$$[z_0 : z_1 : \ldots : z_{2k-2} : z_{2k-1}] \mapsto [-\overline{z_1} : \overline{z_0} : \ldots : -\overline{z_{2k-2}} : \overline{z_{2k-1}}].$$

Suppose that this $\widetilde{F}$ had a fixed point $[z_0 : z_1 : \ldots : z_{2k-2} : z_{2k-1}] \in \mathbb{CP}^{2k-1}$. So there is some $\lambda \in \mathbb{C} - \{0\}$ such that for each $0 \leqslant i \leqslant k-1$, we have $z_{2i} = -\lambda \overline{z_{2i+1}}$ and $z_{2i+1} = \lambda \overline{z_{2i}}$ so $z_{2i} = -\lambda\overline{\lambda} z_{2i} = -|\lambda|^2 z_{2i}$. Since $\lambda \neq 0$, this forces $z_{2i} = 0$ and hence $z_{2i+1} = 0$ for each $0 \leqslant i \leqslant k-1$, a contradiction as not all z_j can be zero. Thus F does not have a fixed point.

Spring 2023-3. Let
$$\omega = \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{(x^2+y^2+z^2)^{3/2}}$$
be a 2-form defined on $\mathbb{R}^3 - \{0\}$ and $S^2 \subset \mathbb{R}^3$ be the unit sphere.

(a) Compute $\int_{S^2} i^*\omega$, where $i : S^2 \to \mathbb{R}^3$ is the inclusion.
(b) Compute $\int_{S^2} j^*\omega$, where $j : S^2 \to \mathbb{R}^3$ is defined by $j(x, y, z) = (2x, 3y, 5z)$.

Hint: Both are 4π. Use Stokes. Show that ω is closed so $\int_E k^*(d\omega) = 0$ where $E = j(S^2) - S^2$ and $k : E \hookrightarrow \mathbb{R}^3 - \{0\}$.

Referenced in: Spring 2015-6.

(a) On S^2, we have $x^2 + y^2 + z^2 = 1$ for any $x, y, z \in S^2$. So
$$i^*\omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy.$$
Hence, we can compute:
$$\begin{aligned}\int_{S^2} i^*\omega &= \int_{S^2} xdy \wedge dz + ydz \wedge dx + zdx \wedge dy \\ &= \int_{D^3} d(xdy \wedge dz + ydz \wedge dx + zdx \wedge dy) \\ &= \int_{D^3} 3dx \wedge dy \wedge dz \\ &= 4\pi,\end{aligned}$$
where the second equality is from Stokes' theorem.

(b) Note first that $j(S^2)$ is an ellipsoid that contains S^2. Let $E = j(S^2) - S^2$. Let $\alpha = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$ and $f = (x^2 + y^2 + z^2)^{-3/2}$ so $\omega = f\alpha$. Then,
$$d\omega = d(f\alpha) = df \wedge \alpha + fd\alpha.$$
As above, we have $d\alpha = 3dV$. Also,
$$df = -(x^2 + y^2 + z^2)^{-5/2}(3xdx + 3ydy + 3zdz),$$
so
$$d\omega = -(x^2 + y^2 + z^2)^{-5/2}(3x^2 + 3y^2 + 3z^2)(dx \wedge dy \wedge dz) + f \cdot 3dV = -3fdV + 3fdV = 0$$
implying that ω is closed. Now, we consider the inclusion $k : E \hookrightarrow \mathbb{R}^3 - \{0\}$. Since $d\omega = 0$, we have
$$0 = \int_E k^*(d\omega) = \int_E dk^*(\omega) = \int_{\partial E} \omega|_{\partial E} = \int_{S^2} j^*\omega - \int_{S^2} i^*\omega$$
by Stokes' theorem, which implies that
$$\int_{S^2} j^*\omega = \int_{S^2} i^*\omega = 4\pi.$$

Spring 2023-4. Let M be a connected compact manifold with non-empty boundary ∂M. Show that M does not retract onto ∂M.

Hint: Use $\mathbb{Z}/2\mathbb{Z}$ coefficients in order to apply Lefschetz duality to simplify the long exact sequence for the pair $(M, \partial M)$. Get $\ker(H_{n-1}(\partial M) \to H_{n-1}(M)) \cong \mathbb{Z}/2\mathbb{Z}$, contradicting the fact that it should be an injection if $r : M \to \partial M$ is a retraction.

Referenced in: Fall 2013-2, Spring 2013-5, Fall 2011-8.

Let $n = \dim(M)$ and note that $(M, \partial M)$ is a good pair so we have the following long exact sequence

$$\cdots \to H_n(\partial M) \to H_n(M) \to H_n(M, \partial M) \xrightarrow{\delta} H_{n-1}(\partial M) \xrightarrow{i_*} H_{n-1}(M) \to H_{n-1}(M, \partial M) \to \cdots .$$

We will work with $\mathbb{Z}/2\mathbb{Z}$ coefficients so that M is orientable and we can apply Lefschetz duality. I.e.,

$$H_n(M; \mathbb{Z}/2\mathbb{Z}) = H^0(M, \partial M; \mathbb{Z}/2\mathbb{Z}) = H_0(M, \partial M; \mathbb{Z}/2\mathbb{Z}) = \widetilde{H}_0(M/\partial M; \mathbb{Z}/2\mathbb{Z}) = 0,$$

also using the universal coefficient theorem and since $M/\partial M$ is connected. On the other hand,

$$H_n(M, \partial M; \mathbb{Z}/2\mathbb{Z}) = H^0(M; \mathbb{Z}/2\mathbb{Z}) = H_0(M; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$$

as M has one connected component. Hence, we have

$$0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{\delta} H_{n-1}(\partial M) \xrightarrow{i_*} H_{n-1}(M).$$

So δ is injective and $\ker(i_*) = \operatorname{im}(\delta) \cong \mathbb{Z}/2\mathbb{Z}$. Now, if $r : M \to \partial M$ is a retraction, then $\mathrm{id}_* = (r \circ i)_* = r_* \circ i_*$ so i_* is injective, a contradiction. Thus, no such retraction exists.

Spring 2023-5. Let $M^m \subset \mathbb{R}^n$ be a closed connected submanifold of dimension m.

(a) Show that $\mathbb{R}^n - M^m$ is connected when $m \leqslant n - 2$.
(b) When $m = n - 1$ show that $\mathbb{R}^n - M^m$ is disconnected by showing that the mod 2 intersection number $I_2(f, M) = 0$ for all smooth maps $f : S^1 \to \mathbb{R}^n$.

Hint: Homotope a path in $\mathbb{R}^n$ to one that is transversal to M and show this must not intersect M. Slice chart with one 0. Take linear path between $(x_1, \ldots, x_m, \varepsilon)$ and $(x_1, \ldots, x_m, -\varepsilon)$ and make into loop to get contradiction.

Referenced in: Fall 2015-6, Fall 2014-2.

(a) Let $x, y \in \mathbb{R}^n - M$. Let $f : [0,1] \to \mathbb{R}^n$ be a path with $f(0) = x$ and $f(1) = y$ and let $S = \{0, 1\} \subset [0, 1]$. By the transversality extension theorem, since $M \subset \mathbb{R}^n$ is closed, $S \subset [0, 1]$ is closed, and $f|_S$ is (vacuously) transverse to M, we know that f is homotopic to a map $g : [0, 1] \to \mathbb{R}^n$ that is transverse to $M \subset \mathbb{R}^n$ and so that f and g agree on S. We claim that $g(t) \notin M$ for all $t \in [0, 1]$ so we could take $g : [0, 1] \to \mathbb{R}^n - M$ a path connecting x and y, as desired.

To see this, suppose $g(t) = z \in M$. Then, using the definition of transversality, we have

$$dg_t(T_t[0, 1]) + T_z M = T_z \mathbb{R}^n \implies \dim(dg_t(T_t[0, 1])) + \dim(M) \geqslant \dim(\mathbb{R}^n).$$

However, $\dim(dg_t(T_t[0, 1])) \leqslant 1, \dim(M) = m$ while $\dim(\mathbb{R}^n) = n$ which is impossible as $m \leqslant n - 2$, showing that there is no $g(t) = z \in M$.

(b) As $\mathbb{R}^n$ is simply connected, we may homotope f to some constant map $g : S^1 \to \mathbb{R}^n$ such that $g(S^1) = p \notin M$. Since mod 2 intersection numbers are invariant under homotopy, we have $I_2(f, M) = I_2(g, M) = 0$ since $g(S^1)$ doesn't intersect M at all.

Now, suppose that $\mathbb{R}^n - M$ is connected. Since M is a closed submanifold of dimension $m = n - 1$, for any $p \in M$, we can find a slice chart for M, namely an open neighborhood U of p with coordinate x such that

$$U \cap M = \{(x_1, \ldots, x_m, x_{m+1}) \in \mathbb{R}^n \mid x_{m+1} = 0\}.$$

Since U is open, there is an $\varepsilon > 0$ so that

$$q = (x_1, \ldots, x_m, \varepsilon), r = (x_1, \ldots, x_m, -\varepsilon) \in U,$$

for some choice of $x_1, \ldots, x_m$. Let $\gamma_0 : [0,1] \to \mathbb{R}^n$ be the path defined by

$$\gamma_0(t) = (x_1, \ldots, x_m, (1-2t)\varepsilon),$$

which goes from q to r. Then $I_2(\gamma_0, M) = 1$ since $\gamma_0^{-1}(M) = \gamma_0^{-1}(U \cap M) = \{0.5\}$. Since $\mathbb{R}^n - M$ is connected, we can find a path γ_1 from r to q that is contained in $\mathbb{R}^n - M$. Thus, $I_2(\gamma_1, M) = 0$. Let $\gamma = \gamma_0 \cdot \gamma_1$ be the concatenation of these two paths which is a loop in $\mathbb{R}^n$. Then, γ has mod 2 intersection equal to

$$I_2(\gamma, M) = I_2(\gamma_0, M) + I_2(\gamma_1, M) = 1,$$

contradicting the fact that any map $S^1 \to \mathbb{R}^n$ has mod 2 intersection 0 with M. Thus $\mathbb{R}^n - M$ is not connected.

Spring 2023-6. (a) If X is a finite CW complex and $\widetilde{X} \to X$ is a path-connected n-fold covering map, then show that the Euler characteristics are related by the formula

$$\chi(\widetilde{X}) = n\chi(X).$$

(b) Let $X = \Sigma_g$ be a closed genus g surface. What path-connected, closed surfaces can cover X?

Hint: Lift characteristic maps, n k-cells for each k-cell in X. Just require $2 - 2k = n(2 - 2g)$.

Referenced in: Fall 2020-9, Spring 2020-9, Spring 2009-11.

(a) Let X be an m-dimensional CW complex and let $\widetilde{X}$ be a path-connected n-fold covering of X. Then, we can lift the CW structure on X to a CW structure on $\widetilde{X}$ by lifting the characteristic maps $\phi_i : D^k \to X$ via the covering $p : \widetilde{X} \to X$ since $\pi_1(D^k) = 0$. There are exactly n lifts of ϕ_i to $\widetilde{X}$ so for each k-cell e_i^k in X, there are n k-cells in the lifted CW structure on $\widetilde{X}$, each of which is mapped homeomorphically down onto e_i^k. Hence, we can calculate

$$\chi(\widetilde{X}) = \sum_{i=0}^{m} (-1)^i \widetilde{C}_i = \sum_{i=0}^{m} (-1)^i n C_i = n \sum_{i=0}^{m} (-1)^i C_i = n\chi(X),$$

where $\widetilde{C}_i, C_i$ are the number of i-cells in $\widetilde{X}, X$ respectively.

(b) Let $\widetilde{X}$ be a path-connected, closed surface covering X. Then, $\widetilde{X}$ is either a sphere or a genus g torus since the cover of an orientable surface is orientable. So $\chi(\widetilde{X}) = 2$ or $2 - 2k$ for some $k \in \mathbb{N}$. Similarly, $\chi(X) = 2 - 2g$ and by part (a), we have $\chi(\widetilde{X}) = n\chi(X)$ for some $n \in \mathbb{N}$. So $\widetilde{X}$ is not a sphere and must have $\chi(\widetilde{X}) = 2 - 2k$. Then, $2 - 2k = n(2 - 2g) \implies 1 - k = n(1 - g) \implies k = 1 + n(g - 1)$. Thus, the only possibilities for $\widetilde{X}$ are the genus k tori for $k = 1 + n(g - 1)$ for $n \in \mathbb{N}$. To see that all of these possibilities do work, we can cut n loops out of $\widetilde{X}$ surrounding a central hole of $\widetilde{X}$ (so there are $n(g - 1)$ other ones) and identify them to get an n-fold covering of X.

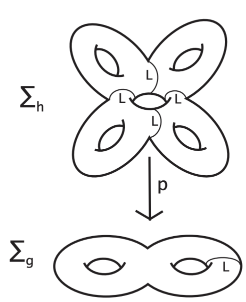


FIGURE 2. Σ_5 covering Σ_2 with 4-fold symmetry

Spring 2023-7. A group G is divisible if for all n, the map $g \mapsto g^n$ from G to itself is surjective. Show that if X is a path-connected CW-complex and if $\pi_1(X, x)$ is a divisible group, then the only path-connected finite cover of X is X itself. (Hint: this can be proven directly or by first showing that a divisible group has no finite index subgroups.)

Hint: Consider action of G on left cosets G/H, homomorphism $G \to S_n$. Then $g \in G - H \implies h^{n!} = g$ for some $h \in G$ a contradiction. Use Galois correspondence.

We show the hint. Let G be a divisible group and suppose $H \subsetneq G$ is a finite index subgroup with $[G : H] = n$. Let G act on the left cosets G/H by left multiplication. This induces a group homomorphism $\psi : G \to S_n$. Then for any $g \in G$, we have $\psi(g^{n!}) = \psi(g)^{n!} = \mathrm{id}_{S_n}$ since S_n is a finite group with order $n!$. Since $H \neq G$, we can find $g' \in G - H$ and since G is divisible we can find $g \in G$ such that $g^{n!} = g'$. But we know $\psi(g') \neq \mathrm{id}_{S_n}$ since $g' \notin H$ while $\psi(g^{n!}) = \mathrm{id}_{S_n}$, a contradiction.

Now, let $p : \widetilde{X} \to X$ be a path-connected finite covering space of X. So $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) \subset \pi_1(X, x_0)$ corresponds to a finite index subgroup of $\pi_1(X, x_0)$. But the only finite index subgroup of $\pi_1(X, x_0)$ is itself so $p_*(\pi_1(\widetilde{X}, x_0)) = \pi_1(X, x_0)$ implying that $\widetilde{X}$ is just X itself.

Spring 2023-8. Let M be an n-manifold, and consider a small disk D^n embedded in M. Show that the inclusion

$$\overline{M - D^n} \hookrightarrow M$$

induces an isomorphism on π_1 if $n \geqslant 3$ and a surjection if $n \geqslant 2$.

Hint: Van Kampen's with $\overline{M - D^n}, D^n$.

First, note that $M = \overline{M - D^n} \cup D^n$ and $S^{n-1} \cong \overline{M - D^n} \cap D^n$. Since S^{n-1} is connected, we can apply Van Kampen's theorem. In particular, we have that

$$f : \pi_1(D^n) * \pi_1(\overline{M - D^n}) \to \pi_1(M)$$

is a surjection. For $n \geqslant 2, D^n$ is simply connected so $\pi_1(D^n)$ is the trivial group so the map $i_* : \pi_1(\overline{M - D^n}) \to \pi_1(M)$ is a surjection for $n \geqslant 2$. We also know by Van Kampen's that we quotient out by the normal subgroup generated by cycles of the intersection to get an isomorphism. However, if $n \geqslant 3$, then S^{n-1} is itself simply connected so has trivial fundamental group and so i_* is indeed an isomorphism.

Spring 2023-9. Find, as a function of n and m, the homology groups

$$H_*(\mathbb{RP}^{n+m}, \mathbb{RP}^n; \mathbb{Z}).$$

Hint: Give $\mathbb{RP}^{n+m}$ a CW structure with one cell in each dimension. Then, relative chain complex is just sequence of $\mathbb{Z}$'s followed by a sequence of 0's. Split into n even/odd cases. Funny business only for $i = n+1$ when n is odd.

Referenced in: Fall 2019-2.

We inductively give $\mathbb{RP}^{n+m}$ a CW structure such that its k-skeleton is $\mathbb{RP}^k$ for all $0 \leqslant k \leqslant n+m$. To do this, we attach a k-cell via the boundary map $\phi_k : S^{k-1} \to \mathbb{RP}^{k-1}$ which is the quotient by the antipodal identification, $x, -x \mapsto [x]$. Then, the chain complex for $\mathbb{RP}^{n+m}$ has $\mathbb{Z}$ in degree k for $0 \leqslant k \leqslant n+m$ and 0 elsewhere while that of $\mathbb{RP}^n$ has $\mathbb{Z}$ in degree k for $0 \leqslant k \leqslant n$ and 0 elsewhere.

Thus, the relative chain complex for the pair $(\mathbb{RP}^{n+m}, \mathbb{RP}^n)$ has $\mathbb{Z}$ in degree k for $n+1 \leqslant k \leqslant n+m$ and we know that the boundary maps $\partial_k : C_k(\mathbb{RP}^{n+m}, \mathbb{RP}^n) \to C_{k-1}(\mathbb{RP}^{n+m}, \mathbb{RP}^n)$ are multiplication by $1 + (-1)^k$

(for $n+2 \leqslant k \leqslant n+m$) since ϕ_k has degree $1+(-1)^k$. Splitting into cases depending on the parity of n and m, we can compute the relative homology to be:

$$H_i(\mathbb{RP}^{n+m}, \mathbb{RP}^n; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i = n+m \text{ and } n+m \text{ is odd, or } i = n+1 \text{ and } n \text{ is odd,} \\ \mathbb{Z}/2\mathbb{Z} & i \text{ odd and } n < i < n+m, \\ 0 & \text{otherwise.} \end{cases}$$

Spring 2023-10. Consider the CW-complexes $A = S^n \vee S^n$, $X = S^n \times S^n$, and $B = S^n \times [0,1]/* \times [0,1]$, where $*$ is the basepoint of S^n. There are inclusions $A \hookrightarrow X$ given by the pairs of points where at least one is the basepoint and $A \hookrightarrow B$ which takes one S^n to $S^n \times 0$ and the other to $S^n \times 1$. Compute the homology of

$$Y = X \cup_A B.$$

Hint: $H_i(Y) = \mathbb{Z}$ for $i = 0, n, 2n$ and 0 otherwise. Use $H_*(A) = \mathbb{Z}^2_{(n)} \oplus \mathbb{Z}_{(0)}$, $H_*(X) = \mathbb{Z}_{(2n)} \oplus \mathbb{Z}^2_{(n)} \oplus \mathbb{Z}^2_{(0)}$, and $H_*(B) = \mathbb{Z}_{(n)} \oplus \mathbb{Z}_{(0)}$ (from Van Kampen's and Künneth) and then use Mayer Vietoris for reduced homology.

Let U and V be neighborhoods of X and B respectively formed by thickening them slightly. So $U \cap V$ deformation retracts to A and we get the following long exact sequence via Mayer Vietoris:

$$\cdots \to H_i(A) \to H_i(X) \oplus H_i(B) \to H_i(Y) \to H_{i-1}(A) \to \cdots.$$

We can compute the homology of A by Van Kampen's theorem to be

$$\widetilde{H}_i(A) = \widetilde{H}_i(S^n) \oplus \widetilde{H}_i(S^n) = \begin{cases} \mathbb{Z}^2 & i = n, \\ 0 & \text{otherwise.} \end{cases}.$$

We can also compute the homology of X, this time using the Künneth theorem:

$$H_i(X) = \bigoplus_{j+k=i} H_j(S^n) \otimes H_k(S^n) = \begin{cases} \mathbb{Z} & i = 2n, \\ \mathbb{Z}^2 & i = 0, n, \\ 0 & \text{otherwise.} \end{cases}$$

Further, we can see that B deformation retracts onto S^n so $H_i(B) = H_i(S^n)$. Now, note that X and B are both path connected so Y is also path connected after we glue X and B together. Thus $H_0(Y) = \mathbb{Z}$. To compute the higher homology, we split into cases: $n = 1$ and $n > 1$.

In the first, we have

$$0 \to H_2(X) \oplus H_2(B) \xrightarrow{f_*} H_2(Y) \xrightarrow{g_*} H_1(A) \xrightarrow{h_*} H_1(X) \oplus H_1(B) \to H_1(Y) \to 0$$

where h_* is induced by the relations on the boundary. By construction, $h_* : \mathbb{Z}^2 \to \mathbb{Z}^3$ is given by $(a,b) \mapsto (a, b, a+b)$ which is injective. Thus, $g_* = 0$ so f_* is an isomorphism (and we know $H_2(B) = 0$), so we get $H_2(X) \cong H_2(Y)$. Thus, $H_2(Y) \cong \mathbb{Z}$. Then,

$$H_1(Y) \cong \operatorname{coker}(h_*) = \mathbb{Z}\langle a, b, c\rangle / \mathbb{Z}\langle a, b, a+b\rangle \cong \mathbb{Z}.$$

In the $n > 1$ case, we have the following two portions of the long exact sequence to consider:

$$0 \to H_{2n}(X) \oplus H_{2n}(B) \to H_{2n}(Y) \to 0$$

and

$$0 \to 0 \to H_{n+1}(Y) \to H_n(A) \xrightarrow{h_*} H_n(X) \oplus H_n(B) \to H_n(Y) \to 0.$$

From the first one, we have $H_{2n}(Y) \cong H_{2n}(X) \oplus H_{2n}(B) \cong \mathbb{Z}$ and from the second, we again have $h_*(a,b) = (a, b, a+b)$ is injective so $H_{n+1}(Y) = 0$ and $H_n(Y) \cong \operatorname{coker}(h_*) \cong \mathbb{Z}$. To summarize, (independent of the case) we have

$$H_i(Y) = \begin{cases} \mathbb{Z} & i = 0, n, 2n, \\ 0 & \text{otherwise.} \end{cases}$$

Fall 2022

Fall 2022-1. The Grassmanian $Gr(k,n)$ is the set of all k-dimensional subspaces of $\mathbb{R}^n$. Explicitly construct the structure of a smooth manifold on $Gr(k,n)$ using atlases. What is its dimension?

Hint: $Gr(k,n) \sim \mathrm{Mat}^k_{n\times k}(\mathbb{R})/\sim$. Define charts on the subsets with each certain $k \times k$ submatrix invertible by reading out the other rows of $AA^{-1}_{i_1,\dots,i_k}$. Dimension is $(n-k)\times k$.

Referenced in: Fall 2008-1.

First, note that we can identify the Grassmanian as

$$Gr(k,n) \sim \mathrm{Mat}^k_{n\times k}(\mathbb{R})/\sim,$$

where $\mathrm{Mat}^k_{n\times k}(\mathbb{R})$ is the space of $n \times k$ real-valued matrices of rank k and $A \sim B$ if and only if there is a matrix $C \in GL_k(\mathbb{R})$ such that $A = BC$. The identification sends a k-dimensional subspace $V \subset \mathbb{R}^n$ to any matrix in $\mathrm{Mat}^k_{n\times k}(\mathbb{R})$ whose image is V (for example, a projection matrix), noting that two matrices are equivalent if and only if they have the same image. This gives a topology on $Gr(k,n)$, namely the quotient topology induced from the standard topology on real matrices.

Now, for each ordered set $1 \leqslant i_1 < \cdots < i_k \leqslant n$, define $U_{i_1,\dots,i_k}$ to be the set of elements $A \in Gr(k,n)$ whose $k \times k$ submatrix $A_{i_1,\dots,i_k}$ is invertible. This is open since this $k \times k$ minor has nonzero determinant and we know that the determinant is continuous. Finally, we define $\varphi_{i_1,\dots,i_k} : U_{i_1,\dots,i_k} \to \mathbb{R}^{(n-k)\times k}$ by sending a matrix A to the entries of the rows of the $(n-k)\times k$ matrix $AA^{-1}_{i_1,\dots,i_k}$ that are complementary to $(i_1,\dots,i_k)$. Note that we will have

$$(AA^{-1}_{i_1,\dots,i_k})_{i_1,\dots,i_k} = I_k,$$

and our maps just read off the rest of the entries. The $\varphi_{i_1,\dots,i_k}$ are clearly homeomorphisms. Smoothness of the transition functions follows from smoothness of the entries of the products and inverses of matrices. Thus, $Gr(k,n)$ has dimension $(n-k)\times k$.

Fall 2022-2. The orthogonal group $O(n)$ is the set of $n \times n$ matrices M satisfying $M^TM = \mathrm{Id}$. Construct the structure of a smooth manifold on $O(n)$ by viewing it as the preimage of a regular value of a smooth map $\mathbb{R}^{n^2} \to \mathbb{R}^{n(n+1)/2}$. Prove that its tangent bundle is trivializable.

Hint: $F : \mathrm{Mat}_{n\times n}(\mathbb{R}) \to \mathrm{Sym}_{n\times n}(\mathbb{R})$ by $M \mapsto M^TM$. Then $dF_A(\frac{1}{2}AC) = C$. Show that all Lie groups are parallelizable by finding a global frame. Take a basis $v_1,\dots,v_n$ for T_eG and define $X_i(g) = dL_g|_e(v_i)$.

Referenced in: Fall 2021-1, Fall 2020-10, Fall 2017-2, Spring 2010-1.

Define $F : \mathrm{Mat}_{n\times n}(\mathbb{R}) \to \mathrm{Sym}_{n\times n}(\mathbb{R})$ by $M \mapsto M^TM$, where $\mathrm{Sym}_{n\times n}(\mathbb{R}) = \{A \in \mathrm{Mat}_{n\times n}(\mathbb{R}) \mid A = A^T\}$ is the set of symmetric matrices. Note that $(M^TM)^T = M^T(M^T)^T = M^TM$ so $F(M)$ is symmetric and F is well-defined. By definition, $O(n) = F^{-1}(\mathrm{Id})$ and we claim that Id is a regular value of F.

Since $\mathrm{Mat}_{n\times n}(\mathbb{R})$ is an $\mathbb{R}$-vector space naturally endowed with a smooth structure, its tangent space is itself and similarly for $\mathrm{Sym}_{n\times n}(\mathbb{R})$. So we can use the standard definition of the differential for vector-valued functions. Namely, we want to show that for any $A \in F^{-1}(\mathrm{Id})$,

$$dF_A : \mathrm{Mat}_{n\times n}(\mathbb{R}) \to \mathrm{Sym}_{n\times n}(\mathbb{R}), \quad B \mapsto \lim_{t\to 0} \frac{F(A+tB)-F(A)}{t}$$

is surjective. For this, let $C \in \mathrm{Sym}_{n\times n}(\mathbb{R})$ be arbitrary and set $B = \frac{1}{2}AC$. Noting that $A^TA = \mathrm{Id}$ since

$F(A) = \text{Id}$ and $C = C^T$ as $C \in \text{Sym}_{n\times n}(\mathbb{R})$, we have

$$\begin{aligned} dF_A(B) &= \lim_{t\to 0} \frac{F(A+tB) - F(A)}{t} = \lim_{t\to 0} \frac{(A+tB)^T(A+tB) - A^TA}{t} \\ &= \lim_{t\to 0} \frac{A^TA + t(B^TA + A^TB) + t^2B^TB - A^TA}{t} \\ &= \lim_{t\to 0}(B^TA + A^TB) + tB^TB = B^TA + A^TB = \frac{(AC)^TA + A^T(AC)}{2} \\ &= \frac{C^TA^TA + A^TAC}{2} = \frac{C^T + C}{2} = C. \end{aligned}$$

It is well-known that $O(n)$ is a Lie group so to show that $O(n)$ has trivializable tangent bundle, we instead show that any lie group G is parallelizable by exhibiting a global frame on G. Choose a basis $v_1, \ldots, v_m$ for T_eG. For each $v_i \in T_eG$, define the vector field X_i by $X_i(g) = dL_g|_e(v_i) \in T_gG$. To see that X_i is smooth, note that $L_g : G \to G$ is smooth in g since $L_g(h) = R_h(g)$ and $R_h : G \to G$ is also smooth for fixed h. Thus, $dL_g|_e : T_eG \to T_gG$ is also smooth. Finally, since L_g is a diffeomorphism, we know that $dL_g|_e$ is a vector space isomorphism for any $g \in G$, showing that $X_1(g), \ldots, X_m(g)$ is a basis for T_gG and $X_1, \ldots, X_m$ is a global frame for G.

Fall 2022-3. Let M be a closed oriented smooth n-manifold. Prove that for every $k \in \mathbb{Z}$, there exists a smooth map $f : M \to S^n$ of degree k.

Hint: Construct $f_k : S^n \to S^n$ of degree k using $z \mapsto z^k$ in S^1 and suspending. $B \subset M$ homeomorphic to $\mathbb{R}^n$, show $q : M \to M/(M-B) \cong S^n$ has degree 1 using a commutative square. Use Whitney's approximation theorem to get smooth map.

Referenced in: Fall 2013-8, Fall 2012-4, Spring 2010-8.

The degree of a map $f : M \to S^n$ is the integer $d \in \mathbb{Z}$ so that $f_* : H_n(M) \to H_n(S^n)$ is multiplication by d. First, define $f_k : S^1 \to S^1$ by $f_k(z) = z^k$. It is easy to see that f_k has degree k. We claim that the suspension of a map preserves its degree. I.e., $\deg(Sf) = \deg(f)$ for any $f : M \to N$. By a standard argument (see Fall 2020-6), we have isomorphisms $\widetilde{H}_i(SM) \cong \widetilde{H}_{i-1}(M)$ for all i. Moreover, this is natural since S is a functor in the sense that we have the following commutative diagram:

$$\begin{array}{ccc} \widetilde{H}_{n+1}(SM) & \xrightarrow{\cong} & \widetilde{H}_n(M) \\ {\scriptstyle Sf_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ \widetilde{H}_{n+1}(SN) & \xrightarrow[\cong]{} & \widetilde{H}_n(N) \end{array}$$

In particular f_* and Sf_* are the same map up to isomorphism so Sf must have the same degree as f. Hence, taking repeated suspensions of the map $f_k : S^1 \to S^1$ gives a map $\widetilde{f}_k : S^n \to S^n$ of degree k.

Now, we construct a map $g : M \to S^n$ of degree 1. To do this, let $B \subset M$ be an open set homeomorphic to an open ball in $\mathbb{R}^n$. Let $q : M \to M/(M-B)$ be the quotient map. Note that $M/(M-B) \cong S^n$. Considering the good pair $(M, M-B)$, the induced long exact sequence is natural so we have a commutative diagram:

$$\begin{array}{ccc} \widetilde{H}_n(M) & \longrightarrow & \widetilde{H}_n(M, M-B) \\ {\scriptstyle q_*}\downarrow & & \downarrow \\ \widetilde{H}_n(M/(M-B)) & \longrightarrow & \widetilde{H}_n(M/(M-B), (M-B)/(M-B)) \end{array}$$

The top map is an isomorphism since M is orientable. The bottom map is an isomorphism since $(M-B)/(M-B)$ is just a single point so has trivial reduced homology. The right map is an isomorphism using excision. Hence q_* is an isomorphism so q has degree 1. Then, let $f = \widetilde{f}_k \circ q$ which has $\deg(f) = \deg(\widetilde{f}_k) \times \deg(q) = k$. Finally, note that degree is (by definition) invariant under homotopy so by Whitney's Approximation theorem, we may homotope f to a smooth map that has the same degree, as desired.

Fall 2022-4. Let M be a smooth manifold and let $\omega \in \Omega^1(M)$ be a nowhere vanishing smooth 1-form. Prove that the following are equivalent.

(a) $\ker(\omega)$ is integrable.
(b) $\omega \wedge d\omega = 0$.
(c) There exists some $\alpha \in \Omega^1(M)$ such that $d\omega = \alpha \wedge \omega$.

Hint: $\ker(\omega) = \operatorname{span}(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^{m-1}}), \omega|_U = f_U dx^m$. Set $\alpha|_U = \frac{df_U}{f_U}$ and put together with partition of unity. For (b) $\implies$ (a), use Frobenius's theorem with $d\omega(X,Y) = -\omega([X,Y])$ to show $[X,Y] \in \ker(\omega)$.

Referenced in: Fall 2018-3, Fall 2018-4, Spring 2015-5, Fall 2014-5, Fall 2013-5, Fall 2008-4.

(a) $\implies$ (c). Let $\ker(\omega)$ be integrable. Then, for every $p \in M$, there exists some chart (U, x) such that, without loss of generality,

$$\ker(\omega) = \operatorname{span}(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^{m-1}})$$

where $m = \dim(M)$ since ω is a 1-form so its kernel is $(m-1)$-dimensional. Then, $\omega|_U = f_U dx^m$ for some nowhere vanishing function $f_U : U \to \mathbb{R}$ so $d\omega|_U = df_U \wedge dx^m$. Choose

$$\alpha_U = \frac{df_U}{f_U} \text{ so that } d\omega|_U = \frac{df_U}{f_U} \wedge f_U dx^m = (\alpha_U \wedge \omega)|_U.$$

Cover M by such charts U and let $\{\phi_U\}$ be a partition of unity subordinate to $\{U\}$ and define

$$\alpha = \sum_U \phi_U \alpha_U.$$

So at every $p \in M$, we have

$$\alpha \wedge \omega = \left(\sum_U \phi_U \alpha_U\right) \wedge \omega = \sum_U \phi_U (\alpha_U \wedge \omega)|_U = \sum_U \phi_U d\omega|_U = d\omega$$

as desired.

(c) $\implies$ (b). Since $d\omega = \alpha \wedge \omega$, we have

$$\omega \wedge d\omega = \omega \wedge (\alpha \wedge \omega) = 0,$$

since $\omega \wedge \omega = 0$ as $\omega \in \Omega^1(M)$ is a 1-form.

(b) $\implies$ (a). Suppose $\omega \wedge d\omega = 0$. We show that $\ker(\omega)$ is involutive which is equivalent to integrable by Frobenius's theorem. Let $X, Y \in \ker(\omega)$. Since ω is a 1-form, we have

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]) = -\omega([X,Y]).$$

Since ω is nowhere vanishing, for any $p \in M$, we can find a $Z \in \mathfrak{X}(M)$ so that $\omega_p(Z) \neq 0$. Hence

$$\begin{aligned} 0 = \omega \wedge d\omega_p(X,Y,Z) &= \omega_p(X) d\omega_p(Y,Z) + \omega_p(Y) d\omega_p(Z,X) + \omega_p(Z) d\omega_p(X,Y) \\ &= \omega_p(Z) d\omega_p(X,Y), \end{aligned}$$

so $d\omega_p(X,Y) = 0$. Thus, we must have $d\omega(X,Y) = 0$ implying that $\omega([X,Y]) = 0$ so $[X,Y] \in \ker(\omega)$ and thus $\ker(\omega)$ is indeed involutive.

Fall 2022-5. Let M be a $2n$-dimensional manifold. A symplectic form on M is a smooth closed 2-form in $\Omega^2(M)$ so that $\omega \wedge \ldots \wedge \omega \in \Omega^{2n}(M)$ is a volume form. (That is, nowhere vanishing). Determine all pairs of positive integers (k, l) so that $S^k \times S^l$ has a symplectic form.

Hint: $(1,1)$ and $(2,2)$ only. Mostly Künneth's formula for the rest. Subtlety for $S^2 \times S^4$ and showing $S^2 \times S^2$ works involving π_i^* for the projection maps and a volume form η on S^2.

We claim the only pairs are $(1,1)$ and $(2,2)$. First, note that $S^k \times S^l$ is closed so for it to have a symplectic form ω, we require ω^n to not be exact. Also $[\omega^n] = [\omega^k] \wedge [\omega^{n-k}]$ so the de Rahm cohomology in any even degree must be nontrivial since the one in the $2n^{\text{th}}$ degree is nontrivial. Moreover, it is clear that $k + l = 2n$ must be even.

Throughout this answer, we use the fact that

$$H^i(S^n) = \begin{cases} \mathbb{Z} & \text{if } i = 0, n, \\ 0 & \text{otherwise.} \end{cases}$$

First, suppose that $k > 2$ and $l > 2$. Then, by Künneth's formula, $H^2(S^k \times S^l) = \bigoplus_{i+j=2} H^i(S^k) \otimes H^j(S^l) = 0$ which means that no such ω exists. Similarly, if $k = 2$ and $l > 4$, we have $H^4(S^2 \times S^l) = 0$. Also by Künneth's formula, we have

$$H^2(S^2 \times S^4) = H^2(S^2) \otimes H^0(S^4) \cong \mathbb{Z}.$$

Consider the projections $\pi_1 : S^2 \times S^4 \to S^2$ and $\pi_2 : S^2 \times S^4 \to S^4$. Since $H^2(S^2 \times S^4) \cong \mathbb{Z}$, it is spanned by $\pi_1^*\eta$ for some volume form η on S^2. If ω was a symplectic form on $S^2 \times S^4$, then $[\omega] = c[\pi_1^*\eta]$ implying that $[\omega^3] = c^3[\pi_1^*\eta^3] = 0$, contradicting the fact that ω^3 is not exact.

Hence, our only choices left are $k = l = 1$ and $k = l = 2$ which we will show are both symplectic. First $S^1 \times S^1$ is an orientable two dimensional manifold (since it is the product of orientable manifolds) and so any volume form is our desired symplectic form. For $S^2 \times S^2$, we again consider the projections $\pi_{1,2} : S^2 \times S^2 \to S^2$. Let η be a volume form on S^2 which exists since S^2 is orientable. Then by Künneth's formula, we have $\pi_1^*\eta \wedge \pi_2^*\eta$ is a volume form on $S^2 \times S^2$. Then, if we take $\omega = \pi_1^*\eta + \pi_2^*\eta$, we have $\omega \wedge \omega = 2\pi_1^*\eta \wedge \pi_2^*\eta$, since $\pi_1^*\eta \wedge \pi_1^*\eta = \pi_1^*(\eta \wedge \eta) = 0$ as $\eta \wedge \eta$ is a 4-form on S^2 and similarly $\pi_2^*\eta \wedge \pi_2^*\eta = 0$ so ω is a closed 2-form with ω^2 a volume form as desired. Note that ω is closed because η is a 2-form on S^2 so is itself closed.

Fall 2022-6. Let C_* be a chain complex of free abelian groups. Let $A_* = C_* \otimes \mathbb{Z}/p$ and let $B_* = C_* \otimes \mathbb{Z}/p^2$ be the chain complexes we get by tensoring C_* degreewise with $\mathbb{Z}/p$ and $\mathbb{Z}/p^2$ respectively.

(a) Show that we have a short exact sequence of chain complexes

$$0 \to A_* \to B_* \to A_* \to 0$$

induced by the corresponding sequences of abelian groups

$$0 \to \mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0.$$

(b) Show how to define a Bockstein natural transformation

$$\beta : H_k(A_*) \to H_{k-1}(A_*)$$

such that we have an associated long exact sequence

$$\cdots \to H_k(A_*) \to H_k(B_*) \to H_k(A_*) \xrightarrow{\beta} H_{k-1}(A_*) \to \cdots$$

(c) Show that if x and y are elements such that $d(x) = py$, then

$$\beta(\overline{x}) = \overline{y},$$

where the bars indicate the reduction modulo p of the corresponding classes.

(d) Show conversely that given an element $\overline{x} \in H_k(A_*)$, if $\beta(\overline{x}) = 0$, then we can find elements $x, y \in C_*$ such that x reduces to $\overline{x}$ modulo p and $d(x) \equiv p^2 y$ modulo p^3.

Hint: $C_* \otimes -$ is an exact functor degreewise. Snake lemma. Follow proof of snake lemma for (c). (d) equivalent to $d(x) \equiv 0 \bmod p^2$.

Referenced in: Fall 2019-9, Spring 2013-10, Spring 2012-5, Spring 2011-6.

(a) This is just because C_* is a chain of free abelian groups (i.e. $\mathbb{Z}^n$) which are all flat so $\mathbb{Z}^n \otimes -$ is an exact functor. Hence $C_* \otimes -$ is degreewise an exact functor giving us the desired short exact sequence of chain complexes.

(b) This is exactly the snake lemma. By part (a), we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_k & \xrightarrow{f} & B_k & \xrightarrow{g} & A_k & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_1} & & \downarrow{\scriptstyle \partial_2} & & \downarrow{\scriptstyle \partial_3} & & \\
0 & \longrightarrow & A_{k-1} & \xrightarrow[f']{} & B_{k-1} & \xrightarrow[g']{} & A_{k-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Let $a \in A_k$ with $\partial_3(a) = 0$. Since g is surjective, there is some $b \in B_k$ with $g(b) = a$. Then, by commutativity of the right square, we have $g' \circ \partial_2(b) = \partial_3 \circ g(b) = \partial_3(a) = 0$ so $\partial_2(b) \in \ker(g')$. By exactness of the second row at B_{k-1}, $\ker(g') = \operatorname{im}(f')$ so there is some $a' \in A_{k-1}$ such that $f'(a') = \partial_2(b)$. In fact, a' is unique since f' is injective. Define $\beta(a) = a'$.

To show that this is well-defined, suppose that $\overline{b} \in B_k$ was some other element with $g(\overline{b}) = a$. Now, $g(\overline{b} - b) = g(\overline{b}) - g(b) = a - a = 0$ so $\overline{b} - b \in \ker(g) = \operatorname{im}(f)$ by exactness of the first row at B_k. Thus, there is some $\overline{a} \in A_k$ so that $f(\overline{a}) = \overline{b} - b$. Again, we find $\overline{a}' \in A_{k-1}$ such that $f'(\overline{a}') = \partial_2(\overline{b})$. By commutativity of the left square, we have $f' \circ \partial_1(\overline{a}) = \partial_2 \circ f(\overline{a}) = \partial_2(\overline{b} - b) = \partial_2(\overline{b}) - \partial_2(b) = f'(\overline{a}') - f'(a') = f'(\overline{a}' - a')$. Now, f' is injective by exactness of the second row so $\partial_1(\overline{a}) = \overline{a}' - a'$ which implies that $\overline{a}' - a' \in \operatorname{im}(\partial_1)$, showing that $\beta : \ker(\partial_3) \to H_{k-1}(A_*) = \frac{\ker(\partial_1)}{\operatorname{im}(\partial_1)}$ is well-defined.

In addition, we need to show that for any $a \in \operatorname{im}(\partial_3)$, we have $\beta(a) = 0 \in H_{k-1}(A_*)$ so that β descends to a map $H_k(A_*) \to H_{k-1}(A_*)$. So let $a \in A_k$ be such that $a = \partial_3(a')$ for some $a' \in A_{k+1}$ and let $b \in B_k$ be such that $g(b) = a$. Then, we can find $b' \in B_{k+1}$ such that $g^{\#}(b') = a'$ where $g^{\#} : B_{k+1} \to A_{k+1}$ is the surjective map in degree $k+1$. Then, consider $b - \partial_2(b') \in B_k$ which we know gets sent to $g(b - \partial_2(b')) = g(b) - g(\partial_2(b')) = a - \partial_3(g^{\#}(b')) = a - \partial_3(a') = a - a = 0$ by g. So then, there exists $\widetilde{a} \in A_k$ such that $f(\widetilde{a}) = b - \partial_2(b')$ and $f'(\partial_1(\widetilde{a})) = \partial_2(f(\widetilde{a})) = \partial_2(b - \partial_2(b')) = \partial_2(b)$ since $\partial_2^2 = 0$. So $\beta(a) = \partial_1(\widetilde{a}) \in \operatorname{im}(\partial_1)$ as desired.

(c) Let $x \in C_k$ and $y \in C_{k-1}$ so that $d(x) = py$. Clearly $\partial_3(\overline{x}) = \overline{py} = 0$ where $\overline{x}$ is reduction of x modulo p. Let $\widehat{x}$ be reduction of x modulo p^2 and we note that $\widehat{x} \in B_k$ satisfies $g(\widehat{x}) = \overline{x}$. Then, $\partial_2(\widehat{x}) = \widehat{py}$ and $\overline{y} \in A_{k-1}$ satisfies $f'(\overline{y}) = \widehat{py}$ so $\beta(\overline{x}) = \overline{y}$ as desired using the definition of β described in part (a).

(d) Suppose $\beta(\overline{x}) = 0$. Let $\widehat{x} \in B_k$ be such that $g(\widehat{x}) = \overline{x}$ and take $\widehat{y} = \partial_2(\widehat{x})$. Then, let $\overline{y} \in A_{k-1}$ be such that $f'(\overline{y}) = \widehat{y}$. In particular, $\overline{y} = \beta(\overline{x}) = 0$ in $\operatorname{coker}(\partial_1)$ so $\overline{y} \in \operatorname{im}(\partial_1)$, say with $\overline{y'} \in A_k$ having $\partial_1(\overline{y'}) = \overline{y}$. Then, $\partial_2(f(\overline{y'})) = f'(\partial_1(\overline{y'})) = f'(\overline{y}) = \widehat{y} = \partial_2(\widehat{x})$. Let $x' \in C_k$ be such that $\widehat{x'} = \widehat{x} - f(\overline{y'})$. Then $\overline{x'} = \overline{\widehat{x} - f(\overline{y'})} = \overline{\widehat{x} - p\overline{y'}} = \overline{x} - 0$ since $\overline{p\overline{y'}} = 0$ and so $\overline{\widehat{x}} = \overline{x}$. In addition, we have $\partial_2(\widehat{x'}) = \partial_2(\widehat{x} - f(\overline{y'})) = 0$ which implies that $d(x') \equiv 0 \bmod p^2$ which suffices to show the result.

Fall 2022-7. Let H be a union of n lines through the origin in $\mathbb{R}^3$. Compute $\pi_1(\mathbb{R}^3 - H)$.

Hint: $*_{1 \leqslant i \leqslant 2n-1}\mathbb{Z}$. Deformation retract to S^2 minus $2n$ points and then homotopy to $\mathbb{R}^2$ minus $2n-1$ points and finally to the wedge sum of $2n-1$ copies of S^1. Use induction/Van Kampen's to prove.

Referenced in: Spring 2021-3, Spring 2016-8.

Note first that $\mathbb{R}^3 - H$ deformation retracts onto $S^2 - J$ where J is a collection of $2n$ points. But we know that $S^2 - \{p\}$ is homotopy equivalent to $\mathbb{R}^2$ so $S^2 - J$ is homotopy equivalent to $\mathbb{R}^2 - K$ where K is a collection of $2n-1$ points. We claim that

$$\pi_1(\mathbb{R}^2 - K) \cong \ast_{1\leqslant i\leqslant 2n-1}\mathbb{Z}$$

is the free product of $2n-1$ copies of $\mathbb{Z}$. We prove this by induction for any collection of k points. The base case $k=0$ is clear, $\pi_1(\mathbb{R}^2) = 0$ and for $k=1$, we have $\mathbb{R}^2 - \{p\}$ deformation retracts to S^1 which has fundamental group $\mathbb{Z}$.

For any collection K of k points, $\mathbb{R}^2 - K$ is homotopy equivalent to $\mathbb{R}^2 - \{(0,0),(1,0),\ldots,(k,0)\}$ so we may assume $K = \{(0,0),(1,0),\ldots,(k,0)\}$. Now, consider the following two open subsets of $\mathbb{R}^2 - K$

$$A = \{(x,y) \in \mathbb{R}^2 - K \mid x < 0.55\}, B = \{(x,y) \in \mathbb{R}^2 - K \mid x > 0.45\}.$$

We have $H_1 \cup H_2 = \mathbb{R}^2 - K$ and $H_1 \cap H_2 = \{(x,y) \in \mathbb{R}^2 \mid 0.45 < x < 0.55\}$ which is simply connected. So, applying Van Kampen's theorem gives us an isomorphism

$$\pi_1(\mathbb{R}^2 - K) \cong \pi_1(A) * \pi_1(B)$$

but A is homotopy equivalent to S^1 since it is homotopy equivalent to $\mathbb{R}^2$ minus a single point so $\pi_1(A) \cong S^1$ while B is homotopy equivalent to $\mathbb{R}^2$ minus $k-1$ points. So by induction, $\pi_1(B) \cong \ast_{1\leqslant i\leqslant k-1}\mathbb{Z}$ and we conclude that $\pi_1(\mathbb{R}^2 - K) \cong \ast_{1\leqslant i\leqslant k}\mathbb{Z}$ as claimed.

Fall 2022-8. Let X be a path connected, locally path connected, semilocally path connected space. Recall that a path connected covering space $\widetilde{X} \to X$ is abelian if $\pi_1(\widetilde{X})$ is normal in $\pi_1(X)$ and the quotient is abelian. Show that there is a universal abelian cover: this is an abelian cover $\widetilde{X} \to X$ such that for any other abelian cover $\widetilde{Y} \to X$, there is a covering map $\widetilde{X} \to \widetilde{Y}$ factoring the map $\widetilde{X} \to X$.

Hint: Universal cover. Galois correspondence gives $p : \widetilde{X} \to X$ with $p_*(\pi_1(\widetilde{X},*)) = [G,G]$. Show the factorization is a cover.

We know that X admits a universal cover $\overline{X} \to X$ since it has all those nice adjectives. Let $\pi_1(X) = G$ and let $H = [G,G] \subset G$ be the commutator subgroup of G. By the Galois correspondence, we know there is a covering space $p : \widetilde{X} \to X$ such that $p_*(\pi_1(\widetilde{X},*)) = H$. We know that H is normal in G and $G/H = G^{\mathrm{ab}}$ is the abelianization of G. I.e., for any normal subgroup $N \subset G$, we have G/N is abelian if and only if $H \subset N$.

Now, if $q : Y \to X$ is an abelian cover, then $q_*(\pi_1(Y,*))$ is normal in G and $G/q_*(\pi_1(Y,*))$ is abelian so

$$p_*(\pi_1(\widetilde{X},*)) \subset q_*(\pi_1(Y,*)).$$

Now, by the lifting property of covering maps, there is some $j : \widetilde{X} \to Y$ such that $p = q \circ j$ which we claim to be a covering map itself. Let $y \in Y$. Then, there is a neighborhood $U \subset X$ of $q(y)$ such that $q^{-1}(U)$ is a disjoint union of open sets that are mapped homeomorphically to U by q since q is a covering map. Also $j^{-1}(q^{-1}(U)) = p^{-1}(U)$ is a disjoint union of open sets that are mapped homeomorphically to U by $q \circ j$. This forces it to be a disjoint union of opens mapped homeomorphically to $q^{-1}(U)$ by j so j is a covering map, completing the proof.

Fall 2022-9. The space $S^1 \times S^1$ is the mapping cone of the map

$$[a,b] : S^1 \to S^1 \vee S^1,$$

representing the commutator of the inclusion of the left summand $a : S^1 \to S^1 \vee S^1$ and the inclusion of the right summand $b : S^1 \to S^1 \vee S^1$. Use this and the long exact sequence to compute the homology.

Hint: General Mayer Vietoris sequence for mapping cylinder and mapping cone. Use knowledge of homology of S^1 and $S^1 \vee S^1$. Use formula to get $[a,b]_* = 0$ so $H_i(S^1 \times S^1) = \mathbb{Z}_{(0)} \oplus \mathbb{Z}^2_{(1)} \oplus \mathbb{Z}_{(2)}$.

Referenced in: Fall 2015-9.

By definition of the mapping cone, we have

$$S^1 \times S^1 = \Big((S^1 \times [0,1]) \bigsqcup (S^1 \vee S^1)\Big) / \sim$$

where $(x,1) \sim [a,b](x)$ for all $x \in S^1$ and $(x,0) \sim (x',0)$ for any $x, x' \in S^1$. We first show that we obtain the following long exact sequence:

$$\cdots \to \widetilde{H}_n(S^1) \xrightarrow{[a,b]_*} \widetilde{H}_n(S^1 \vee S^1) \to \widetilde{H}_n(S^1 \times S^1) \to \cdots .$$

In general, let $f : X \to Y$ be a continuous map between topological spaces, let $A = X \times \{0\} \subset X \times [0,1]$ and define

$$M_f = \Big((X \times [0,1]) \bigsqcup Y\Big) / \sim, \quad C_f = M_f / A,$$

where $(x,1) \sim f(x)$ for all $x \in X$, to be the mapping cylinder and mapping cone of f respectively. Then (M_f, A) forms a good pair since A is closed in M_f and A has a neighborhood, say $X \times [0, 0.5)$, that deformation retracts onto A. Hence, we have $H_i(M_f, A) \cong \widetilde{H}_i(M_f/A) = \widetilde{H}_i(C_f)$ for all i and the long exact sequence for the pair is

$$\cdots \to H_{i+1}(A) \xrightarrow{i_*} H_{i+1}(M_f) \to \widetilde{H}_{i+1}(C_f) \to H_i(A) \xrightarrow{i_*} H_i(M_f) \to \cdots ,$$

where $i : A \hookrightarrow M_f$ is the inclusion. The natural map $X \hookrightarrow A$ is a homeomorphism so $H_i(X) \xrightarrow{\sim} H_i(A)$ for all i where the map on homology takes a chain in X to the same chain in $X \times \{0\} = A$. Moreover, M_f deformation retracts onto Y in a way that provides a homotopy from the inclusion $X \hookrightarrow M_f$ to the map $f : X \to M_f$ given by enlarging the codomain. In particular, if we replace $H_i(A)$ with $H_i(X)$ and $H_i(M_f)$ with $H_i(Y)$, we change i_* to f_* and we get the exact sequence

$$\cdots \to H_{i+1}(X) \xrightarrow{f_*} H_{i+1}(Y) \to \widetilde{H}_{i+1}(C_f) \to H_i(X) \xrightarrow{f_*} H_i(Y) \to \cdots .$$

In our case, this gives us the following long exact sequence

$$\cdots \to \widetilde{H}_n(S^1) \xrightarrow{[a,b]_*} \widetilde{H}_n(S^1 \vee S^1) \to \widetilde{H}_n(S^1 \times S^1) \to \cdots$$

where we know the homologies $\widetilde{H}_*(S^1) = \mathbb{Z}_{(1)}$ and $\widetilde{H}_*(S^1 \vee S^1) = \mathbb{Z}^2_{(1)}$. So this simplifies to

$$0 \to \widetilde{H}_2(S^1 \times S^1) \to \mathbb{Z} \xrightarrow{[a,b]_*} \mathbb{Z}^2 \to \widetilde{H}_1(S^1 \times S^1) \to 0.$$

On first homology, we have $[a,b]_*(x) = (x,0) + (0,x) - (x,0) - (0,x) = (0,0)$ where x is the generator of $\widetilde{H}_1(S^1)$. I.e., $[a,b]_*$ is zero so $\widetilde{H}_2(S^1 \times S^1) \to \mathbb{Z}$ is an isomorphism by exactness, and similarly for $\mathbb{Z}^2 \to \widetilde{H}_1(S^1 \times S^1)$. Finally, $S^1 \times S^1$ is path connected since S^1 and $S^1 \vee S^1$ are, and by the definition of the mapping cone so $H_0(S^1 \times S^1) \cong \mathbb{Z}$. Hence, we conclude

$$H_i(S^1 \times S^1) = \begin{cases} \mathbb{Z} & i = 0, 2, \\ \mathbb{Z}^2 & i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Fall 2022-10. Let $f : X \to Y$ be a continuous, pointed map. Let $\Sigma^n(f) : \Sigma^n X \to \Sigma^n Y$ be the nth (pointed) suspension of f. Show that if for some n, $\Sigma^n(f)$ induces the trivial map on reduced homology, then it does for all n.

Hint: Use naturality of Mayer Vietoris to get a commutative diagram with $\widetilde{H}_k(\Sigma X) \cong \widetilde{H}_{k-1}(X)$.

Referenced in: Fall 2018-9.

For any space X with basepoint $p \in X$, we know that ΣX is defined by $\Sigma X = X \times [0,1]/\sim$ where $(x,0) \sim (x',0)$ and $(x,1) \sim (x',1)$ for any $x, x' \in X$, and $(p,t) \sim (p,t')$ for any $t \in [0,1]$. Let $A = X \times [0,0.55)/\sim$ and $B = X \times (0.45,1]/\sim$, which are both contractible so have trivial reduced homology. We also note that $A \cup B = \Sigma X$ and $A \cap B = X \times (0.45, 0.55)/\sim$ deformation retracts onto just $X \times \{0.5\} \cong X$. Thus, by Mayer Vietoris, we get the following long exact sequence that is natural in X:

$$\cdots \to 0 \to \widetilde{H}_k(\Sigma X) \to \widetilde{H}_{k-1}(X) \to 0 \to \cdots .$$

This naturality exactly means that for any continuous map $f : X \to Y$, we have the following commutative diagram

$$\begin{array}{ccc} \widetilde{H}_k(\Sigma X) & \xrightarrow{\cong} & \widetilde{H}_{k-1}(X) \\ {\scriptstyle (\Sigma f)_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ \widetilde{H}_k(\Sigma Y) & \xrightarrow[\cong]{} & \widetilde{H}_{k-1}(Y) \end{array}$$

So we know that Σf induces the trivial map on reduced homology if and only if f does. Applying the above logic to $\Sigma f : \Sigma X \to \Sigma Y$, we can see that $\Sigma(\Sigma f) = \Sigma^2 f$ induces the trivial map on reduced homology if and only if Σf does. Iterating this, we get the desired result: for any n, $\Sigma^n f$ induces the trivial map on reduced homology if and only if $\Sigma^k f$ does for all k.

Spring 2022

Spring 2022-1. Let M be a closed (compact, without boundary) $2n$-dimensional manifold, and let ω be a closed 2-form on M which is non-degenerate, i.e., for any $p \in M$, the map $T_pM \to T_p^*M$, $X \mapsto i_X\omega(p)$ is an isomorphism. Show that the de Rham cohomology groups $H_{dR}^{2k} \neq 0$ for $0 \leqslant k \leqslant n$.

Hint: Show $\omega(p)$ is a non-degenerate, bilinear, skew-symmetric form. Find basis $X_1, \dots, X_n, Y_1, \dots, Y_n \in T_pM$ with $\omega(p)(X_i, Y_j) = \delta_{ij}$ and then $\omega^n(p)(X_1, \dots, X_n, Y_1, \dots, Y_n) = n!$ implies ω^n vanishes nowhere so M is orientable implies ω^n is not exact since M closed.

It suffices to show $H_{dR}^{2n} \neq 0$ as $[\omega^n] = [\omega^k] \wedge [\omega^{n-k}]$ for any $0 \leqslant k \leqslant n$ so if $[\omega^k] = 0$ for any $0 \leqslant k \leqslant n$, then $[\omega^n] = 0$ as well. Thus, our goal is to show that ω^n is closed but not exact so $[\omega^n] \neq 0 \in H_{dR}^{2n}$. Define

$$f : T_pM \times T_pM \to \mathbb{R}, \quad X \times Y \mapsto \omega(p)(X,Y).$$

f is a bilinear form as $\omega(p)$ is a multilinear map. Also, ω is alternating so

$$\omega(p)(X,Y) = -\omega(p)(Y,X).$$

We now claim that $\omega(p)$ is non-degenerate. Let $X \in T_pM$ and suppose that $\omega(p)(X,Y) = 0$ for all $Y \in T_pM$. I.e., the map $i_X\omega(p) : Y \mapsto \omega(p)(X,Y)$ is exactly the zero map. Then since ω is non-degenerate by assumption, we know that $X = 0$ so $\omega(p)$ is indeed non-degenerate. Thus, $\omega(p)$ is a non-degenerate, bilinear, skew-symmetric form. So there is a basis

$$X_1, \dots, X_n, Y_1, \dots, Y_n \in T_pM$$

so that

$$\omega(p)(X_i, Y_j) = \delta_{ij}, \quad \omega(p)(X_i, X_j) = \omega(p)(Y_i, Y_j) = 0.$$

In particular, we then have

$$\omega(p) = \sum_{i=1}^n X_i^* \wedge Y_i^*$$

and raising to the nth power gives

$$\omega^n(p) = n! X_1^* \wedge \cdots \wedge X_n^* \wedge Y_1^* \wedge \cdots \wedge Y_n^*.$$

Then,

$$\omega^n(p)(X_1, \ldots, X_n, Y_1, \ldots, Y_n) = n!$$

which means that ω^n is nowhere vanishing since $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ is a basis for T_pM so M is orientable. Finally, since ω^n is nowhere vanishing, we have $\int_M \omega^n \neq 0$ so since M is closed, we conclude that ω^n is not exact as claimed.

Spring 2022-2. Let M be a closed n-dimensional manifold. Let ω be a closed k-form on M, $1 \leqslant k \leqslant n$. Prove that for any $p \in M$ there is another closed k-form τ which vanishes on a neighborhood of p and such that $[\omega] = [\tau] \in H^k_{dR}(M)$.

Hint: Ball diffeomorphic to open ball in $\mathbb{R}^n$. Bump function ϕ. $i^*(\omega)$ closed implies exact so find η with $d\eta = i^*(\omega)$. $\tau = \omega - d(\phi\eta)$ works.

Referenced in: Fall 2023-5.

Let $\omega \in \Omega^k(M)$ be closed and let $p \in M$. Let $B \subset M$ be a neighborhood of p that is diffeomorphic to an open ball in $\mathbb{R}^n$ and find $U \subset \overline{U} \subset B$ so that $p \in U$. Let ϕ be a bump function that is supported on B and so that $\phi \equiv 1$ on $\overline{U}$. Let $i : B \hookrightarrow M$ be the inclusion and consider $i^*(\omega)$. $i^*(\omega)$ is closed since $di^*(\omega) = i^*(d\omega) = i^*(0) = 0$. Then, since $H^k(B) \cong H^k(\mathbb{R}^n) = 0$, we know that $i^*(\omega)$ is exact. So we may find $\eta \in \Omega^{k-1}(B)$ such that $d\eta = i^*(\omega)$. Define $\tau = \omega - d(\phi\eta)$. On U, $\phi \equiv 1$ so $\tau = \omega - d\eta = 0$, and clearly by construction $[\omega] = [\tau]$ as they differ by the exact form $d(\phi\eta)$.

Spring 2022- 3. Let M be a closed n-dimensional manifold and let Ω be a volume form (i.e., a nonvanishing n-form) on M. Given a vector field X on M, its divergence $\operatorname{div}(X)$ is the smooth function on M defined by the identity:

$$\mathcal{L}_X(\Omega) = \operatorname{div}(X)\Omega$$

where $\mathcal{L}_X$ denotes the Lie derivative with respect to X.

(a) Prove that $\int_M \operatorname{div}(X)\Omega = 0$.
(b) Express $\operatorname{div}(X)$ in local coordinates.

Hint: Cartan's magic formula, ω is top form, $\partial M = \varnothing$. (b) is $\operatorname{div}(X) = \sum_{i=1}^n \frac{1}{f}\frac{\partial f}{\partial x^i}X^i + \frac{\partial X^i}{\partial x^i}$ for $\Omega = f dx^1 \wedge \cdots \wedge dx^n$.

(a) By Cartan's magic formula, $\operatorname{div}(X)\Omega = \mathcal{L}_X(\Omega) = d\iota_X(\Omega) + \iota_X d\Omega$. But $d\Omega = 0$ since Ω is a top form so

$$\int_M \operatorname{div}(X)\Omega = \int_M d\iota_X(\Omega) = \int_{\partial M} \iota_X(\Omega)$$

by Stokes' theorem. Since M is closed, we have $\partial M = \varnothing$ so $\int_{\partial M} \iota_X(\Omega) = 0$ as desired.

(b) Let $p \in M$ and let (U, x) be a chart centered at p. So, locally we have

$$\Omega = f dx^1 \wedge \cdots \wedge dx^n \text{ and } X = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}$$

for some $f \neq 0$ and $X^i \in \mathbb{R}$. As above, we have $\operatorname{div}(X)\Omega = d\iota_X(\Omega)$ so locally,

$$\operatorname{div}(X)\Omega = d\iota_X(f dx^1 \wedge \cdots \wedge dx^n).$$

Since f is a 0-form,

$$\iota_X(f dx^1 \wedge \cdots \wedge dx^n) = f\iota_X(dx^1 \wedge \cdots \wedge dx^n) + \iota_X f \wedge (dx^1 \wedge \cdots \wedge dx^n) = f\iota_X(dx^1 \wedge \cdots dx^n).$$

Also, we note that

$$\begin{aligned}\iota_X(dx^1 \wedge \cdots \wedge dx^n) &= \sum_{i=1}^{n}(-1)^{i-1}dx^i(X) \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \\ &= \sum_{i=1}^{n}(-1)^{i-1}X^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n.\end{aligned}$$

Thus,

$$d\iota_X(f dx^1 \wedge \cdots \wedge dx^n) = d(f\iota_X(dx^1 \wedge \cdots \wedge dx^n)) = df \wedge \iota_X(dx^1 \wedge \cdots \wedge dx^n) + f d\iota_X(dx^1 \wedge \cdots \wedge dx^n).$$

The first term becomes

$$\sum_{i=1}^{n}(-1)^{i-1}\frac{\partial f}{\partial x^i}X^i dx^i \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n = \sum_{i=1}^{n}\frac{\partial f}{\partial x^i}X^i dx^1 \wedge \cdots \wedge dx^n,$$

and the second term becomes

$$\begin{aligned}fd\left(\sum_{i=1}^{n}(-1)^{i-1}X^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots dx^n\right) =& f\sum_{i-1}^{n}(-1)^{i-1}\frac{\partial X^i}{\partial x^i}dx^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \\ =& \sum_{i=1}^{n} f\frac{\partial X^i}{\partial x^i}dx^1 \wedge \cdots \wedge dx^n.\end{aligned}$$

Putting this together, we have

$$\begin{aligned}\operatorname{div}(X)\Omega &= d\iota_X(f dx^1 \wedge \cdots \wedge dx^n) \\ &= \sum_{i=1}^{n}\frac{\partial f}{\partial x^i}X^i dx^1 \wedge \cdots \wedge dx^n + \sum_{i=1}^{n} f\frac{\partial X^i}{\partial x^i}dx^1 \wedge \cdots \wedge dx^n \\ &= \sum_{i=1}^{n}\left(\frac{\partial f}{\partial x^i}X^i + f\frac{\partial X^i}{\partial x^i}\right)dx^1 \wedge \cdots \wedge dx^n \\ &= \sum_{i=1}^{n}\left(\frac{1}{f}\frac{\partial f}{\partial x^i}X^i + \frac{\partial X^i}{\partial x^i}\right)f dx^1 \wedge \cdots \wedge dx^n,\end{aligned}$$

and so we conclude

$$\operatorname{div}(X) = \sum_{i=1}^{n}\frac{1}{f}\frac{\partial f}{\partial x^i}X^i + \frac{\partial X^i}{\partial x^i}.$$

Spring 2022-4. Let ω be a smooth 1-form on a manifold M and let X and Y be smooth vector fields on M. Use the Cartan formula for Lie derivatives to derive the following formula:

$$d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]).$$

Hint: Use $\mathcal{L}_X = d\iota_X + \iota_X d$ and product rule for $\mathcal{L}_X$ to compute $(\mathcal{L}_X\omega)(Y)$ in two ways and set equal.

Cartan's magic formula is:

$$\mathcal{L}_X = d\iota_X + \iota_X d$$

where X is a vector field. So we directly compute

$$(\mathcal{L}_X\omega)(Y) = d\iota_X\omega(Y) + \iota_X d\omega(Y) = d(\omega(X))(Y) + d\omega(X,Y) = Y(\omega(X)) + d\omega(X,Y),$$

since $\omega(X) : M \to \mathbb{R}$ is a linear functional. We also have the product rule for $\mathcal{L}_X$ which states:

$$(\mathcal{L}_X\omega)(Y) = \mathcal{L}_X(\omega(Y)) - \omega(\mathcal{L}_X(Y)) = X(\omega(Y)) - \omega([X,Y]),$$

as again, $\omega(Y) : M \to \mathbb{R}$ is a linear functional. Combining these two equations gives the result.

Spring 2022-5. Let $N \subset \mathbb{R}^n - \{0\}$ be a compact submanifold of dimension m. Show that N is transverse to almost all k-dimensional linear subspaces in $\mathbb{R}^n$. Here "almost all" means the set of subspaces that are not transverse to N has measure zero.

Hint: Define $F : (\mathbb{R}^k - \{0\}) \times U \to \mathbb{R}^n$ by $(a^1, \ldots, a^k, v_1, \ldots, v_k) \mapsto \sum_i^k a^i v_i$ where $U \subset (\mathbb{R}^n)^k$ is the open set of k-tuples of linearly independent vectors in $\mathbb{R}^n$. Show F is a submersion so is transversal to N and then use Thom's transversality theorem. Surjection $P : U \to Gr(k, n)$ is needed at the end.

Referenced in: Fall 2021-3.

Let $U \subset \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ be the set of k-tuples of linearly independent vectors in $\mathbb{R}^n$. Note that U is open since being linearly independent is an open condition. Define

$$F : (\mathbb{R}^k - \{0\}) \times U \to \mathbb{R}^n, \quad (a^1, \ldots, a^k, v_1, \ldots, v_k) \mapsto \sum_{i=1}^k a^i v_i.$$

Note that the image of F must be contained in $\mathbb{R}^n - \{0\}$ since if $\sum_{i=1}^k a^i v_i = 0$, then all $a^i = 0$ since the v_i are linearly independent. We claim that F is a submersion. Let $(a^1, \ldots, a^k, v_1, \ldots, v_k) \in (\mathbb{R}^k - \{0\}) \times U$ and $w \in \mathbb{R}^n$. There is some $1 \leqslant i \leqslant k$ such that $a^i \neq 0$ so we have

$$\begin{aligned} &dF_{(a^1,\ldots,a^k,v_1,\ldots,v_k)}(0, \ldots, 0, 0, \ldots, \frac{w}{a^i}, \ldots, 0) \\ &= \lim_{t \to 0} \frac{F(a^1, \ldots, a^k, v_1, \ldots, v_i + \frac{w}{a^i}t, \ldots, v_k) - F(a^1, \ldots, a^k, v_1, \ldots, v_k)}{t} \\ &= \lim_{t \to 0} \frac{a^1 v_1 + \ldots a^i(v_i + \frac{w}{a^i}t) + a^k v_k - (a^1 v_1 + \ldots + a^k v_k)}{t} \\ &= \lim_{t \to 0} \frac{wt}{t} = w, \end{aligned}$$

showing that dF is surjective at any point so F is a submersion. Then, F is trivially transverse to any $N \subset \mathbb{R}^n - \{0\}$. By Thom's transversality theorem, for almost all $v = (v_1, \ldots, v_k) \in U$, the map $F_v : \mathbb{R}^k - \{0\} \to \mathbb{R}^n - \{0\}$ is transverse to N. By construction, this is the same as saying that the k-dimensional linear subspace spanned by the set v is transverse to N for almost all linearly independent sets of k vectors. Finally, since there is a surjection $P : U \to Gr(k, n)$ to the space of k-dimensional linear subspaces of $\mathbb{R}^n$, this means that almost every k-dimensional linear subspace of $\mathbb{R}^n$ is transverse to N.

Spring 2022-6. Describe all the connected covering spaces of $\mathbb{RP}^2 \vee \mathbb{RP}^2$. Here $\vee$ is the wedge sum.

Hint: Four cases. Always have $\mathbb{RP}^2$ on the end or loops back or goes to infinity.

Let $X = \mathbb{RP}^2 \vee \mathbb{RP}^2$. We know that the universal cover of $\mathbb{RP}^2$ is S^2 and this is a double cover. Moreover, the fundamental group of $\mathbb{RP}^2$ is $\mathbb{Z}/2\mathbb{Z}$ so (by Van Kampen's for example), $\pi_1(X) = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} = G$ is the group coproduct. So connected coverings of X correspond to subgroups of G.

The trivial subgroup corresponds to the universal cover which is an infinite chain of S^2's. The subgroups isomorphic to $\mathbb{Z}$ are generated by either $(ab)^n$ or $(ba)^n$ and have index $2n$ for some $n \geqslant 1$. These correspond to the "necklace" of $2n$ copies of S^2. The subgroups isomorphic to $\mathbb{Z}/2\mathbb{Z}$ are generated by $(ab)^m a$ or $(ba)^m b$ for some $m \geqslant 0$. These correspond to an $\mathbb{RP}^2$ attached to an infinite chain of S^2's extending in one direction. Finally, there are proper subgroups isomorphic to $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$, generated by $(ab)^n$ and $(ab)^m a$ for $m \leqslant n$. These correspond to a finite chain of S^2's with an $\mathbb{RP}^2$ on both ends.

Spring 2022-7. Let X be a CW complex consisting of one vertex p, 2 edges a and b, and two 2-cells f_1 and f_2, where the boundaries of a and b map to p, the boundary of f_1 is mapped to the loop ab^3 (that is first a and then b thrice), and the boundary of f_2 is mapped to ba^3.

(a) Compute the fundamental group $\pi_1(X)$ of X. Is it a finite group?
(b) Compute the homology groups of X with integer coefficients.

Hint: Group presentation. $\pi_1(X) = \mathbb{Z}/8\mathbb{Z}$. $H_*(X;\mathbb{Z}) = \mathbb{Z}_{(0)} \oplus \mathbb{Z}/8\mathbb{Z}_{(1)}$.

Referenced in: Spring 2015-8.

(a) By the CW construction, we know the fundamental group has presentation

$$\pi_1(X) = \langle a, b \mid ab^3, ba^3 \rangle.$$

We see that $ab^3 = 1 \implies a = b^{-3}$ so $ba^3 = 1 \implies b(b^{-3})^3 = 1 \implies b^8 = 1$. Then, $a = b^{-3} = b^5$ so in fact $\pi_1(X)$ is generated by just b and we have the relation $b^8 = 1$ implying that $\pi_1(X)$ is a quotient group of $\mathbb{Z}/8\mathbb{Z}$. But it is easy to see that we don't have $b^4 = 1$ from the given relations so indeed $\pi_1(X) \cong \mathbb{Z}/8\mathbb{Z}$, a finite group.

(b) Clearly X is path connected so $H_0(X) = \mathbb{Z}$. Also, $H_n(X) = 0$ for $n \geqslant 3$ since X has only simplices in dimensions $\leqslant 2$. Moreover, we know that $H_1(X) = \pi_1(X)^{\mathrm{ab}} = \mathbb{Z}/8\mathbb{Z}$. Hence, we are only concerned with $H_2(X) = \frac{\ker(\partial_2)}{\mathrm{im}(\partial_3)}$ where $C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1$ is a part of the chain complex. We know that $\partial_3 = 0$ since $C_3 = 0$ so we only need to investigate $\partial_2 : C_2 \to C_1$. By the CW definition of X, we know that ∂_2 sends the two generators f_1 and f_2 of C^2 to $a + 3b$ and $b + 3a$ respectively. Suppose $\partial_2(xf_1 + yf_2) = 0$. Then, $x(a+3b) + y(3a+b) = a(x+3y) + b(3x+y) = 0$ so $x + 3y = 0$ and $3x + y = 0$ clearly implying that $x = y = 0$ so ∂_2 is injective and thus $H_2(X) = 0$. To summarize, we have

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0, \\ \mathbb{Z}/8\mathbb{Z} & n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Spring 2022-8. Let X be a topological space and let $Y = (X \times [0,1])/\sim$, where $(x,0) \sim (x',0)$ and $(x,1) \sim (x',1)$ for all $x, x' \in X$. Compute the homology groups of Y in terms of those of X.

Hint: $\widetilde{H}_k(Y) = \widetilde{H}_{k-1}(X)$ for all k. Use Mayer-Vietoris.

This is exactly Fall 2020-6.

Spring 2022-9. Let M be a compact odd-dimensional manifold with nonempty boundary ∂M. Show that the Euler characteristic of M and ∂M are related by $\chi(M) = \frac{1}{2}\chi(\partial M)$.

Hint: Glue two copies of M together along the boundary to make N. N is closed, odd-dimensional so has $\chi(N) = 0$ by Poincaré duality (use $\mathbb{Z}/2\mathbb{Z}$ coefficients) and then use Mayer-Vietoris to get equation.

Referenced in: Spring 2019-10, Spring 2016-4, Fall 2012-8.

Take two copies M_1 and M_2 of M and glue them along the boundary to get the manifold

$$N = M_1 \cup_{\partial M} M_2.$$

Then, N is a closed odd-dimensional (say $2n+1$) manifold. By the definition of the Euler characteristic, we have

$$\chi(N) = \sum_{i=0}^{2n+1} (-1)^i \mathrm{Rank}(H_i(N;F))$$

for any field F. Then, using $F = \mathbb{Z}/2\mathbb{Z}$ so that N is $\mathbb{Z}/2\mathbb{Z}$-orientable, we apply Poincaré duality to get $H_{2n+1-k}(N;\mathbb{Z}/2\mathbb{Z}) \cong H_k(N;\mathbb{Z}/2\mathbb{Z})$ for any $0 \leqslant k \leqslant 2n+1$. Then, $2n+1-k$ and k have different parity so we get

$$\chi(N) = \sum_{i=0}^{n} \big((-1)^i \mathrm{Rank} H_i(N)) + (-1)^{2n+1-i} \mathrm{Rank}(H_{2n+1-i}(N))\big) = 0.$$

Take neighborhoods A and B of ∂M in M_1 and M_2 respectively that can be deformation retracted onto ∂M. Now, we consider the Mayer-Vietoris sequence for $N = (M_1 \cup B) \cup (M_2 \cup A)$ where $(M_1 \cup B) \cap (M_2 \cup A) = A \cap B$ is homeomorphic to ∂M. So we have

$$\cdots \to H_{n+1}(N) \to H_n(\partial M) \to H_n(M_1) \oplus H_n(M_2) \to H_n(N) \to H_{n-1}(\partial M) \to \cdots .$$

Now, we claim that in any exact sequence of finitely generated abelian groups, the alternating sum of the ranks of the groups is equal to 0. Consider

$$0 \xrightarrow{f_{-1}} X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} 0.$$

For each $i \in \{-1, \ldots, n\}$, we have a short exact sequence

$$0 \to \ker(f_i) \to X_i \to \operatorname{im}(f_i) \to 0$$

so $\operatorname{Rank}(X_i) = \operatorname{Rank}(\ker(f_i)) + \operatorname{Rank}(\operatorname{im}(f_i))$ which implies that

$$\sum_{i=-1}^{n} (-1)^i \operatorname{Rank}(X_i) = \sum_{i=-1}^{n} (-1)^i \operatorname{Rank}(\ker(f_i)) + (-1)^i \operatorname{Rank}(\operatorname{im}(f_i)).$$

But since the original sequence was exact, we know that $\ker(f_{i+1}) \cong \operatorname{im}(f_i)$ for all i so then this alternating sum collapses to 0 since all the terms cancel out.

Thus, in our case, we have

$$\sum_{i=0}^{2n+1} (-1)^i \operatorname{Rank}(H_i(N)) - \sum_{i=0}^{2n+1} (-1)^i \operatorname{Rank}(H_i(\partial M)) + \sum_{i=0}^{2n+1} (-1)^i \operatorname{Rank}(H_i(M_1) \oplus H_i(M_2)) = 0$$

which implies that

$$\chi(N) - \chi(\partial M) + 2\chi(M) = 0$$

since $\operatorname{Rank}(H_i(M_1) \oplus H_i(M_2)) = \operatorname{Rank}(H_i(M)) + \operatorname{Rank}(H_i(M))$ as $M_1 = M_2 = M$ so $\chi(M) = \frac{1}{2}\chi(\partial M)$ as desired since $\chi(N) = 0$.

Spring 2022-10. Let $A \in GL(n+1, \mathbb{C})$. It induces a smooth map

$$\phi_A : \mathbb{CP}^n \to \mathbb{CP}^n, \quad [(z_0, \ldots, z_n)] \mapsto [A(z_0, \ldots, z_n)],$$

where $[(z_0, \ldots, z_n)]$ is the usual equivalence class of $(z_0, \ldots, z_n)$ in $(\mathbb{C}^{n+1} - \{0\})/\sim$ where $(z_0, \ldots, z_n) \sim (\lambda z_0, \ldots, \lambda z_n)$, where $\lambda \in \mathbb{C}^\times$. (You do not have to check the smoothness of ϕ_A.)

(a) Show that the fixed points of ϕ_A correspond to eigenvectors of A (up to multiplication by $\mathbb{C}^\times$).
(b) Show that ϕ_A is a Lefschetz map if the eigenvalues of A all have multiplicity 1.
(c) Compute the Euler characteristic of $\mathbb{CP}^n$ be calculating the Lefschetz number of some ϕ_A.

Hint: $(d\phi_A)_x$ has no fixed points for any eigenvector x of A. Use local coordinates. Since we can homotope anywhere by connectedness, use $A = \operatorname{diag}(1, 2, \ldots, n+1)$ to use part (b) so Lefschetz number is sum of local Lefschetz numbers which are all $+1$ so answer is $n+1$.

Referenced in: Spring 2013-6.

(a) Suppose that $\phi_A(x) = x$. Then $[Ax] = [x]$, i.e., $Ax = \lambda x$ for some $\lambda \in \mathbb{C}^\times$ which is exactly what it means for x to be an eigenvector of A (since A is invertible so can't have 0 as an eigenvalue). Conversely, $Ax = \lambda x$ for $\lambda \in \mathbb{C}$ implies that $[\phi_A(x)] = [Ax] = [\lambda x] = [x]$ so indeed x is a fixed point of ϕ_A.

(b) Recall that a map $f : X \to Y$ is a Lefschetz map if $\operatorname{graph}(f) := \{(x, f(x)) \mid x \in X\}$ is transversal to $\Delta = \{(x, x) \mid x \in X\}$. This is equivalent to saying that df_x has no eigenvectors with eigenvalue 1 for all fixed points $x \in X$ of f. I.e., we need only show for any eigenvector x of A, that $(d\phi_A)_x$ has no eigenvectors with eigenvalue 1. Let $\lambda_0, \ldots, \lambda_n$ be the distinct eigenvalues with corresponding eigenbasis $v_0, \ldots, v_n$. Since A is invertible we know that $\lambda_i \neq 0$ for all i. Using local coordinates, we compute

$$(d\phi_A)_x \left(\frac{z_0}{z_i}, \ldots, \widehat{z_i}, \ldots, \frac{z_n}{z_i} \right) = \left(\frac{\lambda_0}{\lambda_i} \frac{z_0}{z_i}, \ldots, \widehat{z_i}, \ldots, \frac{\lambda_n}{\lambda_i} \frac{z_n}{z_i} \right).$$

Then, since all of the eigenvalues are distinct, it is easy to see that $(d\phi_A)_x$ has no eigenvector with eigenvalue 1.

(c) We know that $\chi(\mathbb{CP}^n) = L(\text{id}) = L(\phi_I)$ where $L(f)$ is the Lefschetz number of a map $f : \mathbb{CP}^n \to \mathbb{CP}^n$ and $I \in GL(n+1, \mathbb{C})$ is the identity matrix. We know that the Lefschetz number of a map is invariant under homotopy. Also, since $GL(n+1, \mathbb{C})$ is connected, ϕ_I is homotopic to ϕ_A for any matrix A. Thus, let us take $A = \text{diag}(1, 2, \ldots, n+1)$. Let $e_1, e_2, \ldots, e_{n+1} \in \mathbb{C}^{n+1}$ be the standard basis which is also the eigenvectors for A corresponding to the distinct eigenvalues $1, 2, \ldots, n+1$ respectively. Thus, we have

$$L(\phi_A) = \sum_{i=1}^{n+1} L_{e_i}(\phi_A)$$

since ϕ_A is a Lefschetz map by part (b). We also know that $L_{e_i}(\phi_A) = \pm 1$ depending on whether $d(\phi_A)_{e_i} - I$ is orientation-preserving or orientation-inversing. Since A is a complex matrix, this will always preserve the orientation (of $\mathbb{R}^{2n}$) so we get $L(\phi_A) = \sum_{i=1}^{n+1} 1 = n+1$ and we conclude that $\chi(\mathbb{CP}^n) = n+1$.

Fall 2021

Fall 2021-1. Let $V_k(\mathbb{R}^n)$ denote the space of k-tuples of orthonormal vectors in $\mathbb{R}^n$. Show that $V_k(\mathbb{R}^n)$ is a manifold of dimension $k\left(n - \frac{k+1}{2}\right)$. Hint: Use a map $F : M_{n\times k}(\mathbb{R}) \to \mathbb{R}^{k(k+1)/2}$ such that $V_k(\mathbb{R}^n)$ becomes the preimage of a regular value of F. (Here $M_{n\times k}(\mathbb{R})$ denotes the set of matrices with n rows and k columns.)

Hint: Define $F : M_{n\times k}(\mathbb{R}) \to \text{Sym}_{k\times k}(\mathbb{R}), F(A) = A^T A$, $dF_A(\frac{1}{2}AC) = C$ shows this is a submersion.

We use the same map as in Fall 2022-6 and the proof is the same. Now the dimension is $nk - k(k+1)/2 = k\left(n - \frac{k+1}{2}\right)$.

Fall 2021-2. Show that the product of two spheres $S^p \times S^q$ is parallelizable provided p or q is odd. (Here parallelizable means the tangent bundle is trivializable; equivalently, there exists $(p+q)$ vector fields on $S^p \times S^q$ which are everywhere linearly independent.)

Hint: Nowhere vanishing vector field on $S^{2n-1} : (x_1, \ldots, x_{2n}) \mapsto (-x_2, x_1, -x_4, x_3, \ldots, -x_{2n}, x_{2n-1})$. Gymnastics with vector bundles using projection maps and moving two copies of $\mathbb{R}^2$ over to get trivial bundle.

Referenced in: Spring 2010-2.

Without loss of generality, let $p = 2n-1$ be odd. Define the map

$$X : S^p \to TS^p, \quad (x_1, \ldots, x_{2n}) \mapsto (-x_2, x_1, -x_4, x_3, \ldots, -x_{2n}, x_{2n-1}).$$

In fact, X is a vector field since

$$\begin{aligned} x \cdot X(x) &= (x_1, \ldots, x_{2n}) \cdot (-x_2, x_1, -x_4, x_3, \ldots, -x_{2n}, x_{2n-1}) \\ &= -x_1x_2 + x_2x_1 - x_4x_3 + x_3x_4 - \ldots - x_{2n-1}x_{2n} + x_{2n}x_{2n-1} = 0. \end{aligned}$$

Define a one-dimensional vector bundle on S^p by $B_x = \text{span}(X(x))$. $B \cong S^p \times \mathbb{R}$. We can find the orthogonal complement bundle $B^\perp$ so that $TS^p = B \oplus B^\perp$. Note that the normal bundle NS^p is trivial as S^p has codimension 1 when embedded into $\mathbb{R}^{p+1}$. Finally, we know that for any $x \in S^p$, we have

$$T_x\mathbb{R}^{p+1} = T_xS^p + N_xS^p,$$

and similarly for S^q. So, letting $\pi_1 : S^p \times S^q \to S^p$ and $\pi_2 : S^p \times S^q \to S^q$ be the canonical projections, we have the following chain of isomorphisms:

$$\begin{aligned}
T(S^p \times S^q) &\cong \pi_1^*(TS^p) \oplus \pi_2^*(TS^q) \\
&\cong \pi_1^*(B \oplus B^\perp) \oplus \pi_2^*(TS^q) \\
&\cong \pi_1^*(B^\perp \oplus S^p \times \mathbb{R}) \oplus \pi_2^*(TS^q) \\
&\cong \pi_1^*(B^\perp) \oplus \pi_2^*(TS^q \oplus S^q \times \mathbb{R}) \\
&\cong \pi_1^*(B^\perp) \oplus \pi_2^*(TS^q \oplus NS^q) \\
&\cong \pi_1^*(B^\perp) \oplus \pi_2^*(S^q \times \mathbb{R}^{q+1}) \\
&\cong \pi_1^*(B^\perp \oplus S^p \times \mathbb{R}^2) \oplus S^p \times S^q \times \mathbb{R}^{q-1} \\
&\cong \pi_1^*(TS^p \oplus NS^p) \oplus S^p \times S^q \times \mathbb{R}^{q-1} \\
&\cong S^p \times S^q \times \mathbb{R}^{p+1} \oplus S^p \times S^q \times \mathbb{R}^{q-1} \\
&\cong S^p \times S^q \times \mathbb{R}^{p+q},
\end{aligned}$$

showing that $S^p \times S^q$ is parallelizable.

Fall 2021-3. Let $M^m \subset \mathbb{R}^n - \{0\}$ be a compact smooth submanifold of dimension m. Show that M is transverse to almost all k-dimensional linear subspaces in $\mathbb{R}^n$. (Here "almost all" means that the set of subspaces that are not transverse to M has measure zero.)

Hint: Define $F : (\mathbb{R}^k - \{0\}) \times U \to \mathbb{R}^n$ by $(a^1, \ldots, a^k, v_1, \ldots, v_k) \mapsto \sum_i^k a^i v_i$ where $U \subset (\mathbb{R}^n)^k$ is the open set of k linearly independent vectors in $\mathbb{R}^n$. Show F is a submersion so is transversal to N and then use Thom's transversality theorem. Surjection $P : U \to Gr(k, n)$ is needed at the end.

This is exactly Spring 2022-5.

Fall 2021-4. Let $\omega \in \Omega_c^n(\mathbb{R}^n)$ be a compactly supported n-form. Show that $\omega = d\eta$ for some compactly supported $(n-1)$-form $\eta \in \Omega_c^{n-1}(\mathbb{R}^n)$ if and only if $\int_{\mathbb{R}^n} \omega = 0$.

Hint: Split into $n = 1, n > 1$ cases. Find balls $\text{supp}(\omega) \subset B \subset \overline{B} \subset B'$. Find τ such that $d\tau = \omega$ by Poincaré's lemma, and look at the restriction of τ to $\mathbb{R}^n - \overline{B}$. Show that τ is exact on this domain (using Stokes', closedness, and assumption in question) so get γ on $\mathbb{R}^n - \overline{B}$ and set $\eta = \tau - d(\phi\gamma)$ for ϕ a bump function. Note: need to look at isomorphism on de Rham cohomology induced by the inclusion $S \hookrightarrow \mathbb{R}^n - \overline{B}$ for some sphere S.

If $\omega = d\eta$, then

$$\int_{\mathbb{R}^n} \omega = \int_{\partial \mathbb{R}^n} \eta = 0$$

by Stokes' theorem which we may apply since η has compact support, and since $\partial \mathbb{R}^n = \varnothing$. For the reverse direction, we have two cases to consider.

If $n = 1$, then $\omega = f dx$ for some compactly supported $f \in C^\infty(\mathbb{R})$. Define $F : \mathbb{R} \to \mathbb{R}$ by

$$F(x) = \int_{-\infty}^x f(t) dt.$$

By the fundamental theorem of calculus, $dF = F' dx = f dx = \omega$. Now, since f is compactly supported, there exists some $R > 0$ such that $\text{supp}(f) \subset [-R, R]$. So when $x < -R$, we clearly have $F(x) = 0$. But also, when $x > R$, we have

$$F(x) = \int_{-\infty}^x f(t) dt = \int_{-\infty}^\infty f(t) dt = \int_{\mathbb{R}} \omega = 0,$$

so we have $\text{supp}(F) \subset [-R, R]$ and $dF = \omega$ as desired.

Secondly, if $n > 1$, we instead pick $B, B' \subset \mathbb{R}^n$ two open balls centered at 0 so that

$$\text{supp}(\omega) \subset B \subset \overline{B} \subset B'.$$

Now, ω is a top form so is closed. Hence, by Poincaré's lemma, we know that ω is exact so there is a smooth $(n-1)$-form τ on $\mathbb{R}^n$ such that $d\tau = \omega$. Note, in particular, that $d\tau = 0$ on $\mathbb{R}^n - \overline{B}$.

Consider the restriction of τ to $\mathbb{R}^n - \overline{B}$. By the above, τ is closed on this domain. Moreover, the inclusion $i : \partial\overline{B'} \to \mathbb{R}^n - \overline{B}$ induces an isomorphism $i^* : H^{n-1}_{dR}(\mathbb{R}^n - \overline{B}) \to H^{n-1}_{dR}(\partial\overline{B'})$ where $\partial\overline{B'}$ is an $(n-1)$-sphere contained in $\mathbb{R}^n - \overline{B}$ centered at the origin onto which $\mathbb{R}^n - \overline{B}$ deformation retracts. Then, since τ is closed on $\mathbb{R}^n - \overline{B}$, we know that τ is exact on $\mathbb{R}^n - \overline{B}$ if and only if $i^*\tau$ is exact on $\partial\overline{B'}$ which is true if and only if $\int_{\partial\overline{B'}} i^*\tau = 0$. However, by Stokes' theorem, we have

$$0 = \int_{\mathbb{R}^n} \omega = \int_{\mathbb{R}^n - \overline{B'}} \omega + \int_{\overline{B'}} \omega = 0 + \int_{\overline{B'}} d\tau = \int_{\partial\overline{B'}} i^*\tau$$

so τ is indeed exact on $\mathbb{R}^n - \overline{B}$. So we can find an $(n-2)$-form γ on $\mathbb{R}^n - \overline{B}$ such that $\tau = d\gamma$. Let ϕ be a smooth bump function that is supported on $\mathbb{R}^n - \overline{B}$ and equal to 1 on $\mathbb{R}^n - B'$. So $\eta = \tau - d(\phi\gamma)$ is smooth on all of $\mathbb{R}^n$, compactly supported (because on $\mathbb{R}^n - B'$, $\phi \equiv 1$ and $\eta = \tau - d\gamma = 0$), and $d\eta = d\tau - d^2(\phi\gamma) = d\tau = \omega$ as desired.

Fall 2021-5. Let $n \geqslant 0$ be an integer. Let M be a compact, orientable smooth manifold of dimension $4n+2$. Show that $\dim(H^{2n+1}(M;\mathbb{R}))$ is even.

Hint: $H^{2n+1}(M) \times H^{2n+1}(M) \to \mathbb{R}$ by $(\omega, \eta) \mapsto \int_M \omega \wedge \eta$. Show this is bilinear, anti-symmetric, and non-degenerate. Corresponds to an invertible $k \times k$ ($k = \dim(H^{2n+1}(M))$) matrix with $A = -A^T$ but if k is odd, we get $\det(A) = -\det(A)$, a contradiction.

Referenced in: Spring 2019-10, Spring 2015-10, Fall 2012-7.

Consider the map $H^{2n+1}(M) \times H^{2n+1}(M) \to \mathbb{R}$ where $(\omega, \eta) \mapsto \int_M \omega \wedge \eta$. Now, this map is bilinear since

$$(\omega_1 + \omega_2, \eta) = \int_M (\omega_1 + \omega_2) \wedge \eta = \int_M \omega_1 \wedge \eta + \omega_2 \wedge \eta = \int_M \omega_1 \wedge \eta + \int_M \omega_2 \wedge \eta = (\omega_1, \eta) + (\omega_2, \eta),$$

and similarly for the η component. Also, for any $c \in \mathbb{R}$, we have

$$(c\omega, \eta) = \int_M (c\omega) \wedge \eta = c\int_M \omega \wedge \eta = c(\omega, \eta),$$

and again similarly for the η component. Moreover,

$$\int_M \omega \wedge \eta = -\int_M \eta \wedge \omega,$$

since $\eta, \omega \in \Omega_{2n+1}(M)$ are both odd-dimensional so this form is anti-symmetric. We also know that this is non-degenerate by Poincaré duality and de Rham's theorem since $\mathbb{R}$ is a field. Since M is compact and orientable, $H^{4n+2}(M;\mathbb{R}) \cong \mathbb{R}$ and since $\mathbb{R}$ is a field, we have $H^{2n+1}(M;\mathbb{R}) \cong \mathbb{R}^k$ for $k = \dim(H^{2n+1}(M))$. Thus, we can represent this bilinear form by a matrix $A \in Gl_k(\mathbb{R})$ such that $A = -A^T$. Then, if $\dim(H^{2n+1}(M;\mathbb{R}))$ were odd, we would have

$$\det(A) = \det(A^T) = \det(-A) = (-1)^{\dim(H^{2n+1}(M;\mathbb{R}))}\det(A) = -\det(A),$$

contradicting the fact that A is non-singular. Hence, we must have $\dim(H^{2n+1}(M;\mathbb{R}))$ being even.

Fall 2021-6. Let $f : \mathbb{C} \to \mathbb{C}$ be a nowhere zero continuous function. Prove that there exists a continuous function $g : \mathbb{C} \to \mathbb{C}$ such that $f(z) = e^{g(z)}$ for all $z \in \mathbb{C}$.

Hint: $\exp : \mathbb{C} \to \mathbb{C} - \{0\}$ is a covering map and f lifts to g since $\mathbb{C}$ is contractible.

Since f is nowhere zero, we can think of it as a map $f : \mathbb{C} \to \mathbb{C} - \{0\}$. Now, the map $\exp : \mathbb{C} \to \mathbb{C} - \{0\}$ is a covering map. Moreover $f_*(\pi_1(\mathbb{C}, *)) = f_*(0) = 0 = \exp_*(0) = \exp_*(\pi_1(\mathbb{C}, *))$ since $\mathbb{C}$ is contractible. Thus, there exists a lift $g : \mathbb{C} \to \mathbb{C}$ such that $f = \exp \circ g$.

Fall 2021-7. In this problem, work in either the category of topological manifolds or smooth manifolds (your choice). Let M be an n-manifold. Define its orientation double cover $\widetilde{M}$, and explain its structure as a topological/smooth manifold. Prove that the orientation double cover of $\widetilde{M}$ is always disconnected.

Hint: Topological basis $\widetilde{U}_{\mathcal{O}} = \{(p, \mathcal{O}_p) \mid p \in U, \mathcal{O}_p \text{ orientation of } T_pM \text{ determined by } \mathcal{O}\}$. To show disconnected, show $\widetilde{M}$ is orientable by defining a pointwise orientation and showing it is continuous and then π is an even double cover.

Referenced in: Spring 2021-6, Fall 2017-6.

We will work in the category of smooth manifolds. As a set, we define

$$\widetilde{M} = \{(p, \mathcal{O}_p) \mid p \in M \text{ and } \mathcal{O}_p \text{ is an orientation of } T_pM\}.$$

Define the projection $\pi : \widetilde{M} \to M$ by $(p, \mathcal{O}_p) \mapsto p$. Each tangent space has exactly two orientations so any fiber of this map has cardinality 2. For each pair $(U, \mathcal{O})$ of an open subset U of M and an orientation $\mathcal{O}$ on U, define $\widetilde{U}_{\mathcal{O}} \subset \widetilde{M}$ by

$$\widetilde{U}_{\mathcal{O}} = \{(p, \mathcal{O}_p) \mid p \in U \text{ and } \mathcal{O}_p \text{ is the orientation of } T_pM \text{ determined by } \mathcal{O}\}.$$

We claim that the collection of all subsets of the form $\widetilde{U}_{\mathcal{O}}$ form a basis for the topology on $\widetilde{M}$. Let $(p, \mathcal{O}_p) \in \widetilde{M}$ be a point, let U be an orientable neighborhood of p in M, and let $\mathcal{O}$ be some orientation on U. We may assume $\mathcal{O}_p$ is the same as the orientation of T_pM determined by $\mathcal{O}$ by replacing $\mathcal{O}$ with $-\mathcal{O}$ if necessary. Then, $(p, \mathcal{O}_p) \in \widetilde{U}_{\mathcal{O}}$ so indeed this collection of sets covers $\widetilde{M}$.

Secondly, if $\widetilde{U}_{\mathcal{O}}$ and $\widetilde{U}'_{\mathcal{O}'}$ are two such sets and a point $(p, \mathcal{O}_p) \in \widetilde{U}_{\mathcal{O}} \cap \widetilde{U}'_{\mathcal{O}'}$, then $\mathcal{O}_p$ is determined by the combination $\mathcal{O}, \mathcal{O}'$. Consider V the component of $U \cap U'$ that contains p. Then, the restricted orientation $\mathcal{O}|_V$ and $\mathcal{O}'|_V$ agree at p. Since these two orientations agree at a point, they agree on all of V. Thus $\widetilde{U}_{\mathcal{O}} \cap \widetilde{U}'_{\mathcal{O}'}$ contains the basis set $\widetilde{V}_{\mathcal{O}|_V}$ so we have a topology.

To prove the orientation double cover of $\widetilde{M}$ is always disconnected, we first prove that $\widetilde{M}$ is orientable. Let $\widetilde{p} = (p, \mathcal{O}_p) \in \widetilde{M}$. By definition, $\mathcal{O}_p$ is an orientation on T_pM so we can give $T_{\widetilde{p}}\widetilde{M}$ the unique orientation $\widetilde{\mathcal{O}}_{\widetilde{p}}$ so that $d\pi_{\widetilde{p}} : T_{\widetilde{p}}\widetilde{M} \to T_pM$ sends $\widetilde{\mathcal{O}}_{\widetilde{p}}$ to $\mathcal{O}_p$. This defines a pointwise orientation $\widetilde{\mathcal{O}}$ on $\widetilde{M}$. At each point $\widetilde{p}$ in a basis open subset $\widetilde{U}_{\mathcal{O}}$, the orientation $\widetilde{\mathcal{O}}_{\widetilde{p}}$ agrees with the pullback orientation induced from $(U, \mathcal{O})$ by the restriction $\pi|_{\widetilde{U}_{\mathcal{O}}} : \widetilde{U}_{\mathcal{O}} \to U$ so it is smooth and assembles into a smooth orientation on all of M.

Then, since $\widetilde{M}$ is orientable, it is evenly covered by its orientation double cover as every open orientable subset is evenly covered by $\pi : \widetilde{\widetilde{M}} \to \widetilde{M}$. Thus, $\widetilde{\widetilde{M}}$ has two components and is thus disconnected.

Fall 2021-8. Let M be a connected non-orientable manifold whose fundamental group G is simple (that is, has no non-trivial normal subgroup). Prove that G must be isomorphic to $\mathbb{Z}/2$.

Hint: Orientation double cover is connected and corresponds to index 2 subgroup.

We consider the orientation double cover $p : \widetilde{M} \to M$. Since M is non-orientable, $\widetilde{M}$ is connected, and since $\widetilde{M}$ is a double cover, we have that $p_*(\pi_1(\widetilde{M}, *))$ is an index 2 subgroup of $\pi_1(M, *) = G$. We know all index 2 subgroups are normal so since G is simple, we must have $p_*(\pi_1(\widetilde{M}, *)) = 0$. But then, the only group with a trivial index 2 subgroup is $\mathbb{Z}/2$.

Fall 2021-9. Let X be the quotient of the space $\{0, 1, 2\} \times S^1 \times D^2$ by the relation

$$(0, z_1, z_2) \sim (1, z_1, z_2) \sim (2, z_1, z_2) \quad \forall z_1, z_2 \in S^1.$$

(Here S^1 is the unit circle and D^2 is the unit disk, both inside $\mathbb{R}^2$.) Compute the homology groups of X with integer coefficients.

Hint: Mayer-Vietoris with $A = \{0, 1, 2\} \times S^1 \times \{x \in D^2 \mid |x| > \frac{1}{3}\}/\sim$ and $B = \{0, 1, 2\} \times S^1 \times \{x \in D^2 \mid |x| < \frac{2}{3}\}/\sim$ since they deformation retract onto nice things we know the homology of. $H_*(X) = \mathbb{Z}^2_{(3)} \oplus \mathbb{Z}^2_{(2)} \oplus \mathbb{Z}_{(1)} \oplus \mathbb{Z}_{(0)}$.

Let

$$A = \{0, 1, 2\} \times S^1 \times \{x \in D^2 \mid |x| > \frac{1}{3}\}/\sim, \quad B = \{0, 1, 2\} \times S^1 \times \{x \in D^2 \mid |x| < \frac{2}{3}\}/\sim.$$

Then, A deformation retracts onto $\{0, 1, 2\} \times S^1 \times S^1/\sim = S^1 \times S^1$ while B deformation retracts onto $\{0, 1, 2\} \times S^1$, and $A \cap B$ deformation retracts onto $\{0, 1, 2\} \times S^1 \times S^1$.

We know $H_*(S^1) = \mathbb{Z}_{(0)} \oplus \mathbb{Z}_{(1)}$. Also, by Künneth's formula, we know that $H_*(S^1 \times S^1) = \mathbb{Z}_{(0)} \oplus \mathbb{Z}^2_{(1)} \oplus \mathbb{Z}_{(2)}$. So, we have

$$H_*(A) = \mathbb{Z}_{(0)} \oplus \mathbb{Z}^2_{(1)} \oplus \mathbb{Z}_{(2)},$$
$$H_*(B) = \mathbb{Z}^3_{(0)} \oplus \mathbb{Z}^3_{(1)},$$
$$H_*(A \cap B) = \mathbb{Z}^3_{(0)} \oplus \mathbb{Z}^6_{(1)} \oplus \mathbb{Z}^3_{(2)}.$$

Now, we apply Mayer-Vietoris to get the following long exact sequence:

$$0 \to H_3(X) \to \mathbb{Z}^3 \xrightarrow{f_2} \mathbb{Z} \to H_2(X) \to \mathbb{Z}^6 \xrightarrow{f_1} \mathbb{Z}^2 \oplus \mathbb{Z}^3 \to H_1(X) \to \mathbb{Z}^3 \xrightarrow{f_0} \mathbb{Z} \oplus \mathbb{Z}^3 \to H_0(X) \to 0.$$

We analyze in turn what the connecting maps f_2, f_0, and f_1 do. For $f_2 : H_2(A \cap B) \to H_2(A) \oplus H_2(B)$, we note that the boundary relation identifies each copy of $S^1 \times S^1$ so f_2 sends each generator of $H_2(A \cap B)$ to the generator of $H_2(A)$. I.e., we have

$$f_2(a_1, a_2, a_3) = a_1 + a_2 + a_3$$

so $H_3(X) = \ker(f_2) \cong \mathbb{Z}^2$. Also, since f_2 is surjective, this simplifies the left side of the sequence to

$$0 \to H_2(X) \to \mathbb{Z}^6 \xrightarrow{f_1} \mathbb{Z}^2 \oplus \mathbb{Z}^3 \to H_1(X) \to \mathbb{Z}^3 \xrightarrow{f_0} \mathbb{Z} \oplus \mathbb{Z}^3 \to H_0(X) \to 0.$$

For $f_0 : H_0(A \cap B) \to H_0(A) \oplus H_0(B)$, each generator of $H_0(A \cap B)$ maps to the generator of $H_0(A)$. Also, these three generators each map to unique generators of $H_0(B)$. So we have

$$f_0(a_1, a_2, a_3) = (a_1 + a_2 + a_3, a_1, a_2, a_3)$$

which implies that $H_0(X) = \text{coker}(f_0) \cong \mathbb{Z}$. Also, since f_0 is injective, this further simplifies the exact sequence to

$$0 \to H_2(X) \to \mathbb{Z}^6 \xrightarrow{f_1} \mathbb{Z}^2 \oplus \mathbb{Z}^3 \to H_1(X) \to 0.$$

Finally, for f_1, we have, by similar logic,

$$f_1(a_1, a_2, a_3, b_1, b_2, b_3) = (a_1 + a_2 + a_3, b_1 + b_2 + b_3, a_1, a_2, a_3).$$

So $H_2(X) = \ker(f_2) \cong \mathbb{Z}^2$ and $H_1(X) = \operatorname{coker}(f_2) = \mathbb{Z}$. To summarize, we have

$$H_k(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & k = 0, 1, \\ \mathbb{Z}^2 & k = 2, 3, \\ 0 & \text{otherwise.} \end{cases}$$

Fall 2021-10. Consider the following subsets of $\mathbb{R}^3$

$$\begin{aligned} Z &= \{(0, 0, z) \mid z \in \mathbb{R}\} \\ C_1 &= \{(\cos\theta, \sin\theta, 0) \mid \theta \in \mathbb{R}\} \\ C_2 &= \{(2 + \cos\theta, \sin\theta, 0) \mid \theta \in \mathbb{R}\} \end{aligned}$$

Prove there is no self-homeomorphism on $\mathbb{R}^3$ taking $Z \cup C_1$ to $Z \cup C_2$.

Hint: Fundamental groups of $\mathbb{R}^3 - (Z \cup C_1)$ and $\mathbb{R}^3 - (Z \cup C_2)$ are different.

Referenced in: Spring 2024-10.

Let $X_1 = Z \cup C_1, X_2 = Z \cup C_2$. Then, we can see that X_1 is the unit circle in the xy-plane and a vertical line passing through the center, while X_2 is the same vertical line and the unit circle shifted over such that the line no longer passes through the circle. If $f : \mathbb{R}^3 \to \mathbb{R}^3$ is a homeomorphism that maps X_1 to X_2, then f homeomorphically maps $\mathbb{R}^3 - X_1$ to $\mathbb{R}^3 - X_2$ which induces an isomorphism of fundamental groups.

However, $\mathbb{R}^3 - X_1$ deformation retracts onto a torus, for which we know $\pi_1(\mathbb{R}^3 - X_1) = \mathbb{Z}^2$, while $\mathbb{R}^3 - X_2$ deformation retracts onto $S^2 \vee S^1 \vee S^1$, for which we know

$$\pi_1(\mathbb{R}^3 - X_2) = \pi_1(S^2 \vee S^1 \vee S^1) = \pi_1(S^2) * \pi_1(S^1) * \pi_1(S^1) = \mathbb{Z} * \mathbb{Z},$$

by Van Kampen's theorem. Since these fundamental groups are not isomorphic, we must have that no such f exists, as desired.

Spring 2021

Spring 2021-1. Without using homology groups or homotopy groups, directly derive Brouwer's fixed point theorem (any continuous map $f : D^2 \to D^2$ has a fixed point, where D^2 is the closed 2-disk) from the hairy ball theorem (any continuous vector field on S^2 is somewhere 0).

Hint: Define $w(x) = (1 - x \cdot f(x))x - (1 - x \cdot x)f(x)$ and then $X(x, t) = (-tw(x), x \cdot w(x))$ is a continuous nowhere-zero vector field on S^2.

Suppose that $f : D^2 \to D^2$ is a fixed-point-free continuous map. Define

$$w(x) = (1 - x \cdot f(x))x - (1 - x \cdot x)f(x)$$

for $x = (x_1, x_2) \in D^2$. Since f has no fixed points, we can see that, for any $x \in D^2 - S^1$, the vector $w(x)$ is non-zero and for any $x \in S^1$, we have

$$x \cdot w(x) = 1 - x \cdot f(x) > 0.$$

Now, consider $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$ with coordinates $y = (x, t) = (x_1, x_2, t)$ and define

$$X(x, t) = (-tw(x), x \cdot w(x)).$$

By construction, X is a continuous vector field on $S^2 \subset \mathbb{R}^3$ with $y \cdot X(y) = 0$ for all $y \in S^2$. Moreover, $X(y)$ is nowhere vanishing since if $x \in S^1$, then $x \cdot w(x)$ is non-zero, while if $x \in D^2 - S^1$, then t and $w(x)$ are both non-zero. This contradicts the hairy ball theorem, showing that such an f is impossible, proving Brouwer's fixed point theorem.

Spring 2021-2. Solve the following problems:

(a) Let $F : S^n \to S^n$ be a continuous map. Show that if F has no fixed point, then the degree of the map, $\deg(F) = (-1)^{n+1}$.
(b) Show that if X has S^{2n} as universal covering space, then $\pi_1(X) = \{1\}$ or $\mathbb{Z}_2$.

Hint: Lefschetz trace fixed point formula. Group of deck transformations is isomorphic to π_1 and deck transformation is determined by where it sends a single point so degree is a group embedding from deck transformations to $\mathbb{Z}_2$.

Referenced in: Spring 2013-7, Fall 2010-6.

(a) We use Lefschetz theory. Since F has no fixed points, we know that $L(F) = 0$. But, we know

$$\begin{aligned} 0 = L(F) &= \sum_{i=0}^{n} (-1)^i \mathrm{Tr}(F_* : H_i(S^n) \to H_i(S^n)) \\ &= \mathrm{Tr}(F_* : H_0(S^n) \to H_0(S^n)) + (-1)^n \mathrm{Tr}(F_* : H_n(S^n) \to H_n(S^n)) \\ &= 1 + (-1)^n \deg(F). \end{aligned}$$

So $\deg(F) = (-1)^{n+1}$.

(b) If S^{2n} is the universal covering space of X, then the group of deck transformations of S^{2n} is $G(S^{2n}) \cong \pi_1(X)$. Now, deck transformations are completely determined by where they send a single point. So if a deck transformation $F : S^{2n} \to S^{2n}$ has a fixed point, it is the identity and has degree 1. Otherwise, by part (a), F has degree -1. Now, it is easy to see that degree is a group homomorphism from $G(S^{2n}) \to \{+1, -1\} \cong \mathbb{Z}_2$. We just showed that this has trivial kernel, so $G(S^{2n})$ must be $\{1\}$ or $\mathbb{Z}_2$.

Spring 2021-3. Let $p_1, \ldots, p_n$ be n distinct points in $\mathbb{R}^3$. Calculate the integral homology groups of $\mathbb{R}^3 - \{p_1, \ldots, p_n\}$.

Hint: $H_* = \mathbb{Z}^n_{(2)} \oplus \mathbb{Z}_{(0)}$. Deformation retracts onto the wedge sum of n copies of S^2.

We can easily see that $\mathbb{R}^n - \{p_1, \ldots, p_n\}$ deformation retracts onto the wedge sum of n copies of S^2. So,

$$H_k(\mathbb{R}^3 - \{p_1, \ldots, p_n\}) = H_k(\bigvee_{i=1}^{n} S^2) = \begin{cases} \mathbb{Z} & k = 0, \\ \mathbb{Z}^n & k = n, \\ 0 & \text{otherwise.} \end{cases}$$

Spring 2021-4. Let $\Delta_n^{(k)}$ be the k-dimensional skeleton of the n-simplex Δ_n. Calculate the reduced homology groups $\widetilde{H}_i\left(\Delta_n^{(k)}\right)$ for all values of i, k, n.

Hint: $\widetilde{H}_*(\Delta_n^{(k)}) = \mathbb{Z}_{(k)}^{\binom{n}{k+1}}$ by induction using long exact sequence coming from the mapping cone for inclusion $\Delta_{n-1}^{(k-1)} \hookrightarrow \Delta_{n-1}^{(k)}$.

Referenced in: Fall 2008-7.

We show that

$$\widetilde{H}_i(\Delta_n^{(k)}) \cong \begin{cases} \mathbb{Z}^{\binom{n}{k+1}} & \text{if } i = k, \\ 0 & \text{otherwise,} \end{cases}$$

for all i and for all $0 \leqslant k \leqslant n$ by induction on n. When $n = 0$, we simply have Δ_n is a single point which has trivial reduced homology which aligns with the claim since $\binom{0}{1} = 0$. If $n = 1$, then $\Delta_1^{(1)}$ is a single line and $\Delta_1^{(0)}$ is two points. So $\widetilde{H}_*(\Delta_1^{(0)}) = \mathbb{Z}_{(0)}$ and $\widetilde{H}_*(\Delta_1^{(1)}) = 0$ showing the statement is true for $n = 1$ as well since $\binom{1}{2} = 0$. The base case is done.

Let $n > 1$ and assume the statement is true for $n-1$. Since $\Delta_n^{(0)}$ is the disjoint union of $n+1$ points, we have

$$\widetilde{H}_i(\Delta_n^{(0)}) \cong \begin{cases} \mathbb{Z}^n & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

For $k \geqslant 1$, we know that $\Delta_n^{(k)}$ is the mapping cone of the inclusion $\Delta_{n-1}^{(k-1)} \hookrightarrow \Delta_{n-1}^{(k)}$ which gives us the following long exact sequence

$$\cdots \to \widetilde{H}_i(\Delta_{n-1}^{(k-1)}) \to \widetilde{H}_i(\Delta_{n-1}^{(k)}) \to \widetilde{H}_i(\Delta_n^{(k)}) \to \widetilde{H}_{i-1}(\Delta_{n-1}^{(k-1)}) \to \widetilde{H}_{i-1}(\Delta_{n-1}^{(k)}) \to \cdots .$$

When $i \neq k$, the inductive hypothesis gives $\widetilde{H}_i(\Delta_{n-1}^{(k)}) = 0$ and $\widetilde{H}_{i-1}(\Delta_{n-1}^{(k-1)}) = 0$ so $\widetilde{H}_i(\Delta_n^{(k)}) = 0$. When $i = k$, the inductive hypothesis gives $\widetilde{H}_i(\Delta_{n-1}^{(k-1)}) = 0$ and $\widetilde{H}_{i-1}(\Delta_{n-1}^{(k)}) = 0$ so we have a short exact sequence

$$0 \to \widetilde{H}_k(\Delta_{n-1}^{(k)}) \to \widetilde{H}_k(\Delta_n^{(k)}) \to \widetilde{H}_{k-1}(\Delta_{n-1}^{(k-1)}) \to 0$$

which splits since we are only dealing with free modules so we have

$$\widetilde{H}_k(\Delta_n^{(k)}) \cong \mathbb{Z}^{\binom{n-1}{k+1}} \oplus \mathbb{Z}^{\binom{n-1}{k}} \cong \mathbb{Z}^{\binom{n}{k+1}}$$

using the inductive hypothesis again.

Spring 2021-5. Define the complex projective space $\mathbb{CP}^n$ to be the quotient of $\mathbb{C}^{n+1} - \{0\}$ by the relation $x \sim \lambda x$ for all $\lambda \in \mathbb{C} - \{0\}, x \in \mathbb{C}^{n+1} - \{0\}$. Construct a CW complex structure on $\mathbb{CP}^n$ with no odd-dimensional cells and exactly 1 cell in each even dimension up to $2n$. Calculate the fundamental group and the integral homology groups of $\mathbb{CP}^n$.

Hint: Attach by $\phi_n : e^{2n} \to \mathbb{CP}^n, \quad (z_0, \ldots, z_{n-1}) \mapsto [z_0, \ldots, z_{n-1}, t], \quad t = \sqrt{1 - \sum_{i=0}^{n-1} z_i \overline{z_i}}$. $H_*(\mathbb{CP}^n) = \mathbb{Z}_{(0)} \oplus \mathbb{Z}_{(2)} \oplus \cdots \oplus \mathbb{Z}_{(2n)}$ and $\pi_1(\mathbb{CP}^n) = 0$.

Referenced in: Fall 2023-6, Spring 2012-7, Spring 2011-7, Spring 2009-7, Spring 2008-10.

We define the CW structure inductively. $\mathbb{CP}^0$ is a single point, so it clearly has just one 0-cell and no other cells. Now, given a CW structure on $\mathbb{CP}^{n-1}$ with no odd-dimensional cells and exactly 1 cell in each even dimension up to $2(n-1)$, we construct $\mathbb{CP}^n$ by attaching a single $2n$-cell. To do this, we use the standard inclusion $\mathbb{CP}^{n-1} \subset \mathbb{CP}^n$ given by $[z_0 : \ldots : z_{n-1}] \mapsto [z_0 : \ldots : z_{n-1} : 0]$. Then, any point in $\mathbb{CP}^n - \mathbb{CP}^{n-1}$ can be represented by $[z_0, \ldots, z_{n-1}, t]$ where we may choose $t > 0$ to be the real number $\sqrt{1 - \sum_{i=0}^{n-1} z_i \overline{z_i}}$. Thus, this defines a map from the standard $2n$-cell, $e^{2n} \subset \mathbb{C}^n$,

$$\phi_n : e^{2n} \to \mathbb{CP}^n, \quad (z_0, \ldots, z_{n-1}) \mapsto [z_0, \ldots, z_{n-1}, t], \quad t = \sqrt{1 - \sum_{i=0}^{n-1} z_i \overline{z_i}}.$$

On the boundary ∂e^{2n}, we see that $t = 0$ and so ϕ_n indeed attaches ∂e^{2n} to $\mathbb{CP}^{n-1}$. Now, using the CW structure of $\mathbb{CP}^n$, we have a cell complex

$$0 \to \mathbb{Z} \to 0 \to \mathbb{Z} \to 0 \to \cdots \to \mathbb{Z} \to 0 \to \mathbb{Z},$$

where the first $\mathbb{Z}$ is in degree $2n$ and the last is in degree 0. Clearly, all of the boundary maps must be 0 so the integral homology is the same as the chain groups, i.e.,

$$H_k(\mathbb{CP}^n) = \begin{cases} \mathbb{Z} & 0 \leqslant k \leqslant 2n, k \text{ even}, \\ 0 & \text{otherwise.} \end{cases}$$

The fundamental group has for generators, the 1-cells, and for relations, the attaching maps of the 2-cells. However, there are no 1-cells so $\mathbb{CP}^n$ must have trivial fundamental group.

Spring 2021-6. Define the orientation double cover for any topological manifold. What is the orientation double cover of the real projective plane $\mathbb{RP}^n$?

Hint: Unique two-fold orientable covering space with orientation reversing non-trivial deck transformation. $\mathbb{RP}^n \sqcup \mathbb{RP}^n$ with interchanging copies if n is odd and S^n with antipodal map if n is even.

Referenced in: Fall 2023-9, Spring 2008-8.

The orientation double cover for any topological manifold is the unique two-fold orientable covering space of a topological manifold with orientation reversing non-trivial deck transformation. (Extra: see Fall 2021-7 for details on its construction/topology.)

First, note that $\mathbb{RP}^n$ is the quotient space of S^n by the antipodal map and that

$$H_k(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & k = 0, k = n \text{ and } n \text{ odd}, \\ \mathbb{Z}/2\mathbb{Z} & 1 \leqslant k < n, k \text{ odd}, \\ 0 & \text{otherwise}. \end{cases}$$

Hence, $\mathbb{RP}^n$ is orientable if and only if n is odd. In this case, the orientation double cover is simply $\mathbb{RP}^n \sqcup \mathbb{RP}^n$ where the two copies have opposite orientations. Then, the non-trivial deck transformation which interchanges the two copies of $\mathbb{RP}^n$ is orientation reversing.

If n is even, then the orientation double cover is S^n itself. We know that S^n is a double cover of $\mathbb{RP}^n$ and it is orientable. Also, the non-trivial deck transformation is the antipodal map $A : x \mapsto -x$ which has degree $\deg(A) = (-1)^{n+1} = -1$ when n is even, since A is the composition $n+1$ negations, each of which is an orientation reversing diffeomorphism. Thus, the non-trivial deck transformation is orientation reversing so indeed S^n is the orientation double cover of $\mathbb{RP}^n$.

Spring 2021-7. Show that $S^2 \times S^2$ and the connected sum $\mathbb{CP}^2 \# \mathbb{CP}^2$ are not homotopy equivalent.

Hint: Good pair with gluing sphere S^{n-1}. $H^*(\mathbb{CP}^2\#\mathbb{CP}^2) = H_*(\mathbb{CP}^2\#\mathbb{CP}^2) = \mathbb{Z}_{(4)} \oplus \mathbb{Z}^2_{(2)} \oplus \mathbb{Z}_{(0)}$. Different cup product structure on the generators of $H^2(\mathbb{CP}^2\#\mathbb{CP}^2)$ and $H^2(S^2 \times S^2)$ corresponding to the identity 2×2 matrix and the non-trivial permutation matrix, only one of which is positive definite.

Referenced in: Spring 2018-7.

We will show that these two spaces have different cohomology ring structures. First, we claim that if M and N are two closed oriented manifolds of dimension n, and $X = M\#N$, then

$$\widetilde{H}_i(X) = \begin{cases} \widetilde{H}_i(M) \oplus \widetilde{H}_i(N) & i \neq n, \\ \mathbb{Z} & i = n. \end{cases}$$

Let $A = S^{n-1} \subset X$ be the sphere along which M and N are glued. Note that (X, A) form a good pair with $X/A \cong M \vee N$. The induced long exact sequence for this good pair is

$$\cdots \to \widetilde{H}_i(S^{n-1}) \to \widetilde{H}_i(X) \to \widetilde{H}_i(M \vee N) \to \widetilde{H}_{i-1}(S^{n-1}) \to \cdots$$

We know $\widetilde{H}_i(M \vee N) \cong \widetilde{H}_i(M) \oplus \widetilde{H}_i(N)$, and $\widetilde{H}_i(S^{n-1}) = 0$ for $i \neq n-1$ and $\mathbb{Z}$ for $i = n-1$. So for $i < n-1$, we obtain $\widetilde{H}_i(X) \cong \widetilde{H}_i(M) \oplus \widetilde{H}_i(N)$. Since M and N are closed oriented manifolds, so is X which implies that $\widetilde{H}_n(X) = \widetilde{H}_n(M) = \widetilde{H}_n(N) = \mathbb{Z}$. Then, the top part of the sequence is

$$0 \to \mathbb{Z} \to \mathbb{Z}^2 \to \mathbb{Z} \to \widetilde{H}_{n-1}(X) \to \widetilde{H}_{n-1}(M) \oplus \widetilde{H}_{n-1}(N) \to 0$$

Moreover, $\widetilde{H}_{n-1}(M)$ and $\widetilde{H}_{n-1}(N)$ are torsion-free and thus free so since the alternating sum of the rank of the groups in this exact sequence must be 0, we get $\widetilde{H}_{n-1}(X) \cong \widetilde{H}_{n-1}(M) \oplus \widetilde{H}_{n-1}(N)$ finishing the proof of the claim.

In the case of $\mathbb{CP}^n$, all the homology groups are free so the cohomology groups are just dual to them. I.e., using the above claim, we have

$$H^{4-i}(\mathbb{CP}^2 \# \mathbb{CP}^2) \cong H_i(\mathbb{CP}^2 \# \mathbb{CP}^2) = \begin{cases} \mathbb{Z}^2 & i = 2, \\ \mathbb{Z} & i = 0, 4, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, we note that the generators of the two copies of $H^4(\mathbb{CP}^2)$ are identified after gluing in $H^4(\mathbb{CP}^2 \# \mathbb{CP}^2)$. So if we let a_1, a_2 be the generators of $H^2(\mathbb{CP}^2 \# \mathbb{CP}^2)$, we get

$$\begin{aligned} a_1 \smile a_1 &= a_2 \smile a_2, \\ a_1 \smile a_2 &= 0 = a_2 \smile a_1, \end{aligned}$$

where $a_1 \smile a_1 = a_2 \smile a_2$ generates $H^4(\mathbb{CP}^2 \# \mathbb{CP}^2)$. This has corresponding matrix I_2, the 2×2 identity matrix which is positive definite. On the other hand, letting θ be a volume form in S^2, we note that $\pi_1^*(\theta)$ and $\pi_2^*(\theta)$ generate $H^2(S^2 \times S^2)$, and the Künneth formula says

$$\begin{aligned} &\pi_1^*(\theta) \smile \pi_1^*(\theta) = 0, \\ &\pi_1^*(\theta) \smile \pi_2^*(\theta) = (-1)^4(\pi_2^*(\theta) \smile \pi_1^*(\theta)) = \pi_2^*(\theta) \smile \pi_1^*(\theta), \\ &\pi_2^*(\theta) \smile \pi_2^*(\theta) = 0, \end{aligned}$$

where $\pi_1^*(\theta) \smile \pi_2^*(\theta) = \pi_2^*(\theta) \smile \pi_1^*(\theta)$ generates $H^4(S^2 \times S^2)$. This has corresponding matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which is not positive definite since it has both a positive and negative eigenvalue. Hence, the two cup product structures are not the same so the cohomology rings are not isomorphic.

Spring 2021-8. Consider a differentiable map $f : S^{2n-1} \to S^n$ with $n \geqslant 2$. If $\alpha \in \Omega^n(S^n)$ is a differential form of degree n such that $\int_{S^n} \alpha = 1$, let $f^*(\alpha) \in \Omega^n(S^{2n-1})$ be its pull-back under f.

(a) Show that there exists $\beta \in \Omega^{n-1}(S^{2n-1})$ such that $f^*(\alpha) = d\beta$.
(b) Show that the integral $I(f) = \int_{S^{2n-1}} \beta \wedge d\beta$ is independent of the choices of β and α.

Hint: $n \geqslant 2 \implies n < 2n-1 \implies H^n(S^{2n-1}) = 0$ *so closed implies exact. Stokes' theorem.* $H^n(S^n) = \mathbb{Z}$ *so* $\alpha = \alpha' + d\eta$ *for* $\eta \in \Omega^{n-1}(S^n)$. $\eta \wedge d\eta = 0$. *Gymnastics with wedge product and differentials.*

(a) Since α is a top form, $d\alpha = 0$ so α is closed. Then,

$$df^*(\alpha) = f^*(d\alpha) = f^*(0) = 0,$$

so $f^*(\alpha)$ is closed. Since $n \geqslant 2$, we have $n < 2n - 1$ so $H^n(S^{2n-1}) = 0$, implying that $f^*(\alpha)$ is exact. Hence, there exists some $\beta \in \Omega^{n-1}(S^{2n-1})$ such that $f^*(\alpha) = d\beta$.

(b) First, suppose $\beta, \beta' \in \Omega^{n-1}(S^{2n-1})$ both satisfy $d\beta = d\beta' = f^*(\alpha)$. Then, $d(\beta' - \beta) = f^*(\alpha) - f^*(\alpha) = 0$ so $\beta' - \beta$ is closed. Again, $n - 1 < 2n - 1$ so $H^{n-1}(S^{2n-1}) = 0$ and so $\beta' - \beta$ is exact. I.e., we can find $\gamma \in \Omega^{n-2}(S^{2n-1})$ such that $\beta' - \beta = d\gamma$. Then, we have

$$\begin{aligned} \int_{S^{2n-1}} \beta' \wedge d\beta' &= \int_{S^{2n-1}} (\beta + d\gamma) \wedge d(\beta + d\gamma) \\ &= \int_{S^{2n-1}} \beta \wedge d\beta + \int_{S^{2n-1}} d\gamma \wedge d\beta \\ &= \int_{S^{2n-1}} \beta \wedge d\beta + \int_{S^{2n-1}} d(\gamma \wedge d\beta) \\ &= \int_{S^{2n-1}} \beta \wedge d\beta + \int_{\partial S^{2n-1}} \gamma \wedge d\beta = \int_{S^{2n-1}} \beta \wedge d\beta \end{aligned}$$

where the second-last equality comes from Stokes' theorem and the last since S^{2n-1} has empty boundary. Hence, $I(f)$ is independent of the choice of β.

Second, suppose $\alpha, \alpha' \in \Omega^n(S^n)$ both satisfy $\int_{S^n} \alpha = \int_{S^n} \alpha' = 1$. Thus by de Rham's theorem, α and α' are in the same cohomology class since $\int_{S^n} \alpha - \alpha' = 0$ so we can find $\eta \in \Omega^{n-1}(S^n)$ such that $\alpha = \alpha' + d\eta$. Now, let $\beta, \beta' \in \Omega^{n-1}(S^{2n-1})$ be such that $d\beta = f^*(\alpha)$ and $d\beta' = f^*(\alpha')$. So we have

$$d\beta = f^*(\alpha) = f^*(\alpha' + d\eta) = f^*(\alpha) + df^*(\eta) = d\beta' + d\eta' = d(\beta' + \eta')$$

for some $\eta' = f^*(\eta) \in \Omega^{n-1}(S^{2n-1})$. Hence, $\beta - \beta' - \eta'$ is closed, so by similar logic to above so $\beta = \beta' + \eta' + d\gamma$ for some $\gamma \in \Omega^{n-2}(S^{2n-1})$. Thus, we have

$$\begin{aligned}
\int_{S^{2n-1}} \beta \wedge d\beta - \beta' \wedge d\beta' &= \int_{S^{2n-1}} (\beta' + \eta' + d\gamma) \wedge (d\beta' + d\eta') - \beta' \wedge d\beta' \\
&= \int_{S^{2n-1}} \beta' \wedge d\eta' + \eta' \wedge d\beta' + \eta' \wedge d\eta' + d\gamma \wedge (d\beta' + d\eta') \\
&= \int_{S^{2n-1}} \beta' \wedge d\eta' + \eta' \wedge d\beta' + f^*(\eta) \wedge df^*(\eta) + \int_{S^{2n-1}} d(\gamma \wedge (d\beta' + d\eta')) \\
&= \int_{S^{2n-1}} \beta' \wedge d\eta' + d\beta' \wedge \eta' + f^*(\eta) \wedge f^*(d\eta) + \int_{\partial S^{2n-1}} \gamma \wedge (d\beta' + d\eta') \\
&= \int_{S^{2n-1}} d(\beta' \wedge \eta') + f^*(\eta \wedge d\eta) + 0 \quad (*) \\
&= \int_{\partial S^{2n-1}} \beta' \wedge \eta' + \int_{S^{2n-1}} f^*(\eta \wedge d\eta) \\
&= 0 + \int_{S^{2n-1}} f^*(\eta \wedge d\eta) = \int_{S^{2n-1}} f^*(0) = 0,
\end{aligned}$$

as desired, where the second-last equality comes from the fact that $\eta \wedge d\eta$ is a $n - 1 + n = 2n - 1$-form on S^{n-1} so must be 0. Note that $(*)$ only holds when $n-1$ is even (since $d(\beta' \wedge \eta') = d\beta' \wedge \eta' + (-1)^{n-1}\beta' \wedge d\eta'$), but I am pretty sure that the question is just wrong if $n - 1$ is odd.

Spring 2021-9. Let $f : M \to N$ be a smooth map between smooth manifolds, X and Y be smooth vector fields on M and N, respectively, and suppose that $f_*X = Y$ (i.e., $f_*(X(x)) = Y(f(x))$ for all $x \in M$). Then prove that

$$f^*(L_Y\omega) = L_X(f^*\omega).$$

Hint: Show $f^*\iota_Y\omega = \iota_X f^*\omega$ for any $\omega \in \Omega^k(M)$. Cartan's magic formula and $df^* = f^*d$.

We first show $f^*\iota_Y\omega = \iota_X f^*\omega$ for any $\omega \in \Omega^k(M)$. let $E_1, \ldots, E_{k-1}$ be vector fields on M. Then,

$$\begin{aligned}
f^*\iota_Y\omega(E_1, \ldots, E_{k-1}) &= \iota_Y\omega(f_*E_1, \ldots, f_*E_{k-1}) \\
&= \omega(Y, f_*E_1, \ldots, f_*E_{k-1}) \\
&= \omega(f_*X, f_*E_1, \ldots, f_*E_{k-1}) \\
&= f^*\omega(X, E_1, \ldots, E_{k-1}) \\
&= \iota_X f^*\omega(E_1, \ldots, E_{k-1}).
\end{aligned}$$

Then, since $df^* = f^*d$, Cartan's magic formula gives

$$\begin{aligned}
f^*(L_Y\omega) = f^*((d\iota_Y + \iota_Y d)\omega) &= f^*d\iota_Y\omega + f^*\iota_Y d\omega = df^*\iota_Y\omega + f^*\iota_Y d\omega = d\iota_X f^*\omega + \iota_X f^*d\omega \\
&= d\iota_X f^*\omega + \iota_X df^*\omega = (d\iota_X + \iota_X d)(f^*\omega) = L_X(f^*\omega).
\end{aligned}$$

Spring 2021-10. Prove Cartan's lemma: Let M be a smooth manifold of dimension n. Fix $1 \leqslant k \leqslant n$. Let ω^i and φ_i be 1-forms on M. Suppose that the $\{\omega^1, \dots, \omega^k\}$ are linearly independent and that $\sum_{i=1}^k \varphi_i \wedge \omega^i = 0$. Prove that there exist smooth functions $h_{ij} = h_{ji} : M \to \mathbb{R}$ such that for all $i = 1, \dots, k$, $\varphi_i = \sum_{j=1}^k h_{ij}\omega^j$.

Hint: $\alpha_1, \dots, \alpha_r$ linearly dependent if and only if $\alpha_1 \wedge \cdots \wedge \alpha_r = 0$. Look at $(\omega^1 \wedge \cdots \wedge \widehat{\omega}^i \wedge \cdots \wedge \omega^k) \wedge \left(\sum_{j=1}^k \varphi_j \wedge \omega^j\right) = 0$. To show equality, do the same thing but removing both ω^i and ω^j.

By Spring 2014-4, we know prove that $\{\alpha_1, \dots, \alpha_r\}$ is a linearly dependent set of 1-forms if and only if $\alpha_1 \wedge \cdots \wedge \alpha_r = 0$. Now, fix some $1 \leqslant i \leqslant k$. by linearity of the wedge product, we have

$$(\omega^1 \wedge \cdots \wedge \widehat{\omega}^i \wedge \cdots \wedge \omega^k) \wedge \left(\sum_{j=1}^k \varphi_j \wedge \omega^j\right) = (\omega^1 \wedge \cdots \wedge \widehat{\omega}^i \wedge \cdots \wedge \omega^k) \wedge 0 = 0.$$

However, expanding the left-hand side gives

$$\omega^1 \wedge \cdots \wedge \widehat{\omega}^i \wedge \cdots \wedge \omega^k \wedge \varphi_i \wedge \omega^i = \pm\omega^1 \wedge \cdots \wedge \omega^k \wedge \varphi_i,$$

using the fact that $\omega^j \wedge \omega^j = 0$ as ω^j is a 1-form. So we must have $\omega^1 \wedge \cdots \wedge \omega^k \wedge \varphi_i = 0$ which implies that $\{\omega^1, \dots, \omega^k, \varphi_i\}$ is a linearly dependent set so we may write

$$\varphi_i = \sum_{j=1}^k h_{ij}\omega^j$$

for some smooth functions h_{ij}. We now claim that $h_{ij} = h_{ji}$. Without loss of generality, fix $1 \leqslant i < j \leqslant k$. Then, we have

$$\begin{aligned}
0 &= \left(\sum_{l=1}^k \varphi_l \wedge \omega^l\right) \wedge \omega^\wedge \cdots \wedge \widehat{\omega}^i \wedge \cdots \wedge \widehat{\omega}^j \wedge \cdots \wedge \omega^k \\
&= \varphi_i \wedge \omega^i \wedge \omega^1 \wedge \cdots \wedge \widehat{\omega}^i \wedge \cdots \wedge \widehat{\omega}^j \wedge \cdots \wedge \omega^k + \varphi_j \wedge \omega^j \wedge \omega^1 \wedge \cdots \wedge \widehat{\omega}^i \wedge \cdots \wedge \widehat{\omega}^j \wedge \cdots \wedge \omega^k \\
&= h_{ij}\omega^j \wedge \omega^i \wedge \omega^1 \wedge \cdots \wedge \widehat{\omega}^i \wedge \cdots \wedge \widehat{\omega}^j \wedge \cdots \wedge \omega^k + h_{ji}\omega^i \wedge \omega^j \wedge \omega^1 \wedge \cdots \wedge \widehat{\omega}^i \wedge \cdots \wedge \widehat{\omega}^j \wedge \cdots \wedge \omega^k.
\end{aligned}$$

Hence, we must have

$$h_{ij}\omega^1 \wedge \cdots \wedge \omega^k = h_{ji}\omega^1 \wedge \cdots \wedge \omega^k.$$

Since $\{\omega^1, \dots, \omega^k\}$ is linearly independent, $\omega^1 \wedge \cdots \wedge \omega^k \neq 0$ so $h_{ij} = h_{ji}$ as claimed.

Fall 2020

Fall 2020-1. Let $x_1, x_2, \dots, x_k$ and $y_1, y_2, \dots, y_k$ be two sets of distinct points in a connected smooth manifold M with $\dim(M) > 1$, and $v_1, v_2, \dots, v_k$ and $w_1, w_2, \dots, w_k$ be the corresponding two sets of non-zero tangent vectors at these points. Show that there is a diffeomorphism f of M such that $f(x_i) = y_i$ and $df_{x_i}(v_i) = w_i$ for $i = 1, \dots, k$.

Hint: Do each $x_i \mapsto y_i$ separately and compose. Connectedness is required. Map v to w in $\mathbb{R}^n$ $(n \geqslant 2)$ using a rotation of the plane spanned by v and w. Need to use flows and 1-parameter subgroup of diffeomorphisms.

Referenced in: Spring 2017-1, Fall 2010-1, Spring 2009-3, Fall 2008-3.

Claim: given $y \in \mathbb{R}^n, r > |y|$, we can find a diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ with compact support in $B(0,r)$ and $f(0) = y$. Let X be the constant vector field that is always y and let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a smooth bump function with $\varphi \equiv 1$ on $B(0, \frac{r+|y|}{2})$ and $\varphi \equiv 0$ on $\mathbb{R}^n - B(0,r)$. Let $\widetilde{X} = \varphi X$, which is a vector field with compact support in $B(0,r)$ and is equal to X on $B(0, \frac{r+|y|}{2})$. Let $\widetilde{\Phi}_t$ be the flow of $\widetilde{X}$. Then $f = \widetilde{\Phi}_1$ is such a diffeomorphism.

Claim: if $\gamma : [0,1] \to M$ is a smooth injective path joining x and y and U is an open neighborhood of γ, then there is a diffeomorphism $f : M \to M$ with compact support in U and $f(x) = y$. By compactness of $\gamma([0,1])$, we can cover the path γ with a finite number of charts, $(U_i, \phi_i)_{0 \leqslant i \leqslant l}$ centered at x_i such that $x_0 = x, x_l = y$, and $x_{i+1} \in U_i$. Then, for $0 \leqslant i \leqslant l-1$, the above claim gives a diffeomorphism $f_i : \mathbb{R}^n \to \mathbb{R}^n$ with compact support such that $f_i(0) = \phi_i(x_{i+1})$. Let $\widetilde{f}_i = \phi_i^{-1} \circ f_i \circ \phi_i$, extended by the identity outside of U_i. Then $\widetilde{f}_i : M \to M$ is a diffeomorphism with compact support in U such that $\widetilde{f}_i(x_i) = x_{i+1}$. It follows that $f = \widetilde{f}_{l-1} \circ \cdots \circ \widetilde{f}_0$ is the desired diffeomorphism.

Claim: we can find a diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ with compact support in $B(0,r)$ such that $f(0) = 0$ and $df_0(v) = w$ for any $v, w \in \mathbb{R}^n - \{0\}$ and $r > 0$. Let $\lambda = \frac{||w||}{||v||} > 0$. If v and w are colinear with $w = \lambda v$, then we can choose $\Phi_t = \lambda^t \mathrm{Id}_{\mathbb{R}^n}$. If not, then let P be the plane spanned by v and w oriented so that $\{v, w\}$ is a direct basis. Let θ be the oriented angle between v and w. If $w = -\lambda v$, take a third vector which is not colinear to v, say w', and consider the plane P spanned by v, w' and let $\theta = \pi$. Then, define R_t to be the linear isomorphism of $\mathbb{R}^n$ such that its restriction to P is rotation by an angle of $t\theta$ and its restriction to $P^\perp$ is the identity map. Let $\Phi_t = \lambda^t R_t$.

In both cases, we have Φ_t a 1-parameter subgroup of diffeomorphisms. Let X be its infinitesimal generator and let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a bump function with $\varphi \equiv 1$ on $B(0, \frac{r}{2})$ and $\varphi \equiv 0$ on $\mathbb{R}^n - B(0,r)$. Define a vector field $\widetilde{X}$ by $\widetilde{X} = \varphi X$ and let $\widetilde{\Phi}_t$ be its flow. We see that $f = \widetilde{\Phi}_1$ is our desired diffeomorphism.

Claim: we prove the question in the case where $k = 1$ where we let U be a connected open neighborhood of $\{x, y\}$ in M. If $x = y$, then choose a chart (V, ϕ) so that $V \subset U$ and $\phi : U \to \mathbb{R}^n$ is a diffeomorphism with $\phi(x) = 0$. By the third claim, we can find a diffeomorphism $\widetilde{f} : \mathbb{R}^n \to \mathbb{R}^n$ with $\widetilde{f}(0) = 0$ and $d\widetilde{f}_0(d\phi_x(v)) = d\phi_x(w)$. Then $f = \phi^{-1} \circ \widetilde{f} \circ \phi$, extended by the identity outside of V is our desired diffeomorphism.

If $x \neq y$, let $\gamma : [0,1] \to M$ be a smooth path joining x and y that lies in U. By the second claim, we can find a diffeomorphism $f_1 : M \to M$ with compact support in U sending x to y. Let $w' = (df_1)_x(v) \in T_y M$. By the first case, we can then find a diffeomorphism $f_2 : M \to M$ with compact support in U such that $f_2(y) = y$ and $(df_2)_y(w') = w$ so $f = f_2 \circ f_1$ is as desired.

Claim: given points $x_1, \ldots, x_k, y_1, \ldots, y_k \in M$, we can find pairwise disjoint, connected, open subsets $U_1, \ldots, U_k$ with $x_i, y_i \in U_i$. Take tubular neighborhoods of paths that join x_i and y_i and the result follows by induction.

Finally, find by the fifth claim, connected open subsets $U_1, \ldots, U_k$ that are pairwise disjoint and with $x_i, y_i \in U_i$. Then for each i, the fourth claim yields a diffeomorphism $f_i : M \to M$ with compact support in U_i such that $f_i(x_i) = y_i$ and $df_{x_i}(v_i) = w_i$. Then $f = f_1 \circ \cdots \circ f_k$ is the desired diffeomorphism.

Fall 2020-2. Let M be a smooth manifold of dimension n. Let $T^*M := \bigsqcup_{m \in M} T_m^* M$ be the cotangent bundle, where $T_m^* M$ is the dual of the tangent space $T_m M$, and let $\pi : T^*M \to M$ be the natural projection such that $\pi(\phi) = m$ for $\phi \in T_m^* M$. Let $x = (x_1, \ldots, x_n)$ be local coordinates on $U \subset M$. Then we endow $\pi^{-1}(U)$ with coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$, such that the element $\phi \in \pi^{-1}(U)$ with $\pi(\phi) = m$ is written as $\sum_i y_i dx_i(m)$.

(a) Show that T^*M is a smooth manifold with respect to the local coordinate charts defined above.
(b) Define the 1-form λ on the cotangent bundle T^*M as follows: for any tangent vector $v \in T_\phi(T^*M)$ at $\phi \in T^*M$, we set $\lambda(\phi)(v) = \phi(d\pi(v))$. Find the explicit expression of λ with respect to the above local coordinate chart. Use this to show that λ is smooth.
(c) Find the explicit expression of $d\lambda$ and its k-th exterior powers for all $k \geqslant 2$ with respect to the local coordinate chart above. Use this to show that T^*M is orientable.

Hint: $\lambda = \sum_i y_i dx_i$ and $(d\lambda)^k = \sum_{1 \leqslant i_1 < \ldots < i_k \leqslant n} k! dy_{i_1} \wedge dx_{i_1} \wedge \cdots \wedge dy_{i_k} \wedge dx_{i_k}$

(a) We give T^*M the topology induced from setting $(x_1, \ldots, x_n, dx_1, \ldots, dx_n) : T^*U \to x(U) \times \mathbb{R}^n$ to be a homeomorphism. This is Hausdorff since covectors above different points are separated by the topology of M which is Hausdorff while covectors above the same point are separated by the Hausdorff topology on $\mathbb{R}^n$ which is induced upon T^*M. The transition functions between charts is the pairing of the transition function for M and the transpose of the inverse of the differential of this transition function which we know to both be smooth so is smooth itself.

(b) We compute $\lambda(\phi)$ by computing what it does to basis vectors, $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial y_i}$. But we note that $d\pi(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial x_i}$ and $d\pi(\frac{\partial}{\partial y_i}) = 0$. Thus,

$$\lambda(\phi)\left(\frac{\partial}{\partial y_i}\right) = \phi\left(d\pi\left(\frac{\partial}{\partial y_i}\right)\right) = \phi(0) = 0$$

and

$$\lambda(\phi)\left(\frac{\partial}{\partial x_i}\right) = \phi\left(d\pi\left(\frac{\partial}{\partial x_i}\right)\right) = \phi\left(\frac{\partial}{\partial x_i}\right) = y_i.$$

So we have

$$\lambda = \sum_i y_i dx_i$$

which is clearly smooth as its component functions are linear with respect to these coordinates.

(c) From part (b), we have

$$d\lambda = d\sum_i y_i dx_i = \sum_i dy_i \wedge dx_i.$$

So,

$$(d\lambda)^k = \sum_{1 \leqslant i_1 < \ldots < i_k \leqslant n} k! dy_{i_1} \wedge dx_{i_1} \wedge \cdots \wedge dy_{i_k} \wedge dx_{i_k}.$$

In particular, $(d\lambda)^n = n! dy_1 \wedge dx_1 \wedge \cdots \wedge dy_n \wedge dx_n$ is a nowhere vanishing top-form (i.e., a volume form) of T^*M so T^*M is orientable.

Fall 2020-3. Let M be a smooth manifold with smooth boundary ∂M and N be a smooth manifold without boundary. Assume that $f : M \to N$ is smooth (this includes smoothness at points of ∂M) so that the tangent map $df_x : T_xM \to T_{f(x)}N$ is well-defined (including at points of ∂M). Let $y \in N$ be a regular value for both f and $f|_{\partial M}$.

(a) Show that $M_1 := f^{-1}(y)$, if not empty, is a smooth submanifold with boundary in M such that the boundary $\partial M_1 = (f_{\partial M})^{-1}(y) = M_1 \cap \partial M$ is a submanifold of ∂M.
(b) If we only assume that y is a regular value for f but not for $f_{\partial M}$, does the conclusion of (a) still hold?

Hint: Use standard preimage theorem. Switch to local coordinates. Show $\ker(df_0) \neq \ker(d(f|_{\partial M})_0)$ to show that $\pi : f^{-1}(0) \cap B \to \mathbb{R}, x \mapsto x_m$ has 0 as a regular value. $\{s \in S \mid \pi(s) \geqslant 0\}$ is manifold with boundary $\{s \in S \mid \pi(s) = 0\}$. Counterexample: $M = \{y \geqslant 0\} \subset \mathbb{R}^2, f : (x, y) \mapsto y \in \mathbb{R}$.

(a) Since $\operatorname{Int}(M)$ is a smooth manifold without boundary, so too is $M_1 \cap \operatorname{Int}(M) = (f|_{\operatorname{Int}(M)})^{-1}(y)$ by the standard preimage theorem. So it suffices to look at M_1 around a point $x \in \partial M$. By using local coordinates, we can treat a neighborhood of y in N as a ball in $\mathbb{R}^n$, with $y = 0$ and we can treat a neighborhood of x in M as an open subset of $H^m = \{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_m \geqslant 0\}$ with $x = 0$. Since f is smooth at 0, we may shrink the neighborhood around x to assume that f is defined and smooth in a ball B around 0.

Let $S = f^{-1}(0) \cap B$. By the preimage theorem, S is a smooth manifold of codimension 1. Define $\pi : S \to \mathbb{R}$ by $x \mapsto x_m$. We see that $B \cap M_1 = \pi^{-1}[0, \infty]$. We claim that 0 is a regular value of π. To see this, note that 0 is a regular value of both f and $f|_{\partial M}$ so both df_0 and $(df|_{\partial M})_0$ are surjective and thus their kernels have the same codimension.

Then, these kernels have different dimensions as they lie in $T_0\partial M$ and T_0M respectively which have different dimensions. So $\ker(df_0) \neq \ker(d(f|_{\partial M})_0)$. Now, $T_0S = \ker(df_0)$ cannot lie in $T_0\partial M$ entirely since $df_0|_{T_0\partial M} =$

$d(f|_{\partial M})_0$. So locally there is a vector $v \in T_0 S$ with $v \notin T_0 \partial M$, i.e., $v_m \neq 0$. Then, $d\pi_0(v) = v_m \neq 0$ so $d\pi_0$ is surjective showing that 0 is a regular value of π.

Now, we claim that if S is a manifold without boundary and $\pi : S \to \mathbb{R}$ is a smooth function with regular value 0, then $\{s \in S \mid \pi(s) \geqslant 0\}$ is a manifold with boundary, whose boundary is $\pi^{-1}(0)$, completing the proof. This is because $\{s \in S \mid \pi(s) > 0\}$ is open in S and is therefore a submanifold of S with the same dimension as S. Since 0 is a regular value, π is locally equivalent to the canonical submersion near 0 and this claim is clear when π is the canonical submersion.

(b) Counterexample: Let $M = \{(x, y) \in \mathbb{R}^2 \mid y \geqslant 0\} \subset \mathbb{R}^2$, $N = \mathbb{R}$, and $f : M \to N$ be given by $(x, y) \mapsto y$. Now, f is linear and $df = f$ so f is a submersion. $y = 0$ is a regular value of f but $\partial M = \{(x, y) \in \mathbb{R}^2 \mid y = 0\}$ and $f|_{\partial M} = 0$ so 0 is not a regular value of $f|_{\partial M}$. Using the above notation, $M_1 = f^{-1}(0) = \{(x, y) \in \mathbb{R}^2 \mid y = 0\} = \partial M$ which is a manifold without boundary $\partial M_1 = \varnothing$. But $M_1 \cap \partial M = \partial M \neq \varnothing$, a contradiction.

Fall 2020-4. Let S be a closed subset of a smooth manifold M that has a second countable topological basis. Show that for any positive integer n, there is a smooth map $f : M \to \mathbb{R}^n$ such that $S = f^{-1}(0)$.

Hint: Do for $\mathbb{R}^n$ using $\phi_{x_0,r}(x) = \exp((|x - x_0|^2 - r^2)^{-1})$ inside $B(x_0, r)$ and 0 outside and paracompactness. Use this to do it for $A \cap U_\alpha$ for a locally finite covering by charts and add to get desired function.

This is done as part of Fall 2016-1.

Fall 2020-5. Let $M, N \subset \mathbb{R}^{p+1}$ be two compact, smooth, oriented submanifolds (without boundary) of dimensions m and n, respectively, such that $m + n = p$, and suppose that $M \cap N = \varnothing$. Let $l(M, N)$ be the degree of the map

$$\lambda : M \times N \to S^p, \quad \lambda(x, y) = \frac{x - y}{||x - y||}.$$

(a) Show that $l(M, N) = (-1)^{(m+1)(n+1)} l(N, M)$.
(b) Show that if M is the boundary of an oriented submanifold $W \subset \mathbb{R}^{p+1}$ which is disjoint from N, then $l(M, N) = 0$.

Hint: $l(N, M) = \deg(\lambda'), \lambda' = A \circ \lambda_M \circ S : N \times M \to S^p$. Define $F(w, n) = \frac{w-n}{||w-n||}$ so that $\lambda = \partial F = F \circ i_M$ so $d\lambda^* \omega = dF^* \omega$ and use Stokes' to integrate $\deg(\lambda) \int_{S^p} \omega$ for any volume form ω.

Referenced in: Fall 2013-3.

(a) Let $S : N \times M \to M \times N$ be the map $(a, b) \mapsto (b, a)$ and let $A : S^p \to S^p$ by the antipodal map. Define $\lambda' = A \circ \lambda_M \circ S : N \times M \to S^p$. As the composition of $p + 1$ reflections, the map A has degree $(-1)^{p+1} = (-1)^{m+n+1}$. Similarly, S is the composition of $m \cdot n$ flips (switching of coordinates) so has degree $(-1)^{mn}$. Note that $l(N, M)$ is the degree of λ' by construction. Thus,

$$l(N, M) = \deg(\lambda') = \deg(A \circ \lambda \circ S) = \deg(A) \deg(\lambda) \deg(S) = (-1)^{(m+1)(n+1)} l(M, N).$$

(b) Let F be the map $F : W \times N \to S^p$ given by

$$F(w, n) = \frac{w - n}{||w - n||},$$

which is well-defined since $W \cap N = \varnothing$. By construction, $\lambda = F \circ i_M$ where $i_M : M \times N \to W \times N$ is the inclusion map since $\partial(W \times N) = \partial W \times N = M \times N$ as N has no boundary. Then, let ω be a volume form in S^p, we have, via Stokes' theorem:

$$\begin{aligned} \deg(\lambda) \int_{S^p} \omega = \int_{M \times N} \lambda^* \omega &= \int_{M \times N} i_M^* F^* \omega \\ &= \int_{W \times N} dF^* \omega = \int_{W \times N} F^*(d\omega) = \int_{W \times N} F^*(0) = \int_{W \times N} 0 = 0, \end{aligned}$$

where $d\omega = 0$ since ω is a top form. Thus $\deg(\lambda) = 0$ as $\int_{S^p} \omega \neq 0$ since ω is a volume form.

Fall 2020-6. Let X be a topological space and let $Y = X \times [-1, 1]/\sim$, where

$$(x, -1) \sim (x', -1) \quad \text{for all } x, x' \in X$$

$$(x, 1) \sim (x', 1) \quad \text{for all } x, x' \in X$$

Describe the relationship between the homology groups of X and Y.

Hint: $\widetilde{H}_k(Y) = \widetilde{H}_{k-1}(X)$ for all k. Use Mayer-Vietoris.

Referenced in: Fall 2022-3, Fall 2022-10, Spring 2022-8, Spring 2016-9, Spring 2014-10.

Note that $Y = S(X)$ is the suspension of X. Take A to be the image of $X \times [-1, 0.1)$ and B to be the image of $X \times (-0.1, 1]$ in $S(X)$. Then A and B are contractible and cover $S(X)$, and $A \cap B$ deformation retracts onto a copy of X. The Mayer-Vietoris sequence for reduced homology is

$$\cdots \to \widetilde{H}_{i+1}(A) \oplus \widetilde{H}_{i+1}(B) \to \widetilde{H}_{i+1}(A \cup B) \to \widetilde{H}_i(A \cap B) \to \widetilde{H}_i(A) \oplus \widetilde{H}_i(B) \to \cdots$$

which in this case is just

$$\cdots \to 0 \to \widetilde{H}_{i+1}(S(X)) \to \widetilde{H}_i(X) \to 0 \to \cdots$$

since A and B are contractible and $A \cap B$ is homotopic to X. Thus we must have $\widetilde{H}_{i+1}(S(X)) \cong \widetilde{H}_i(X)$ for all i (by convention $\widetilde{H}_{-1}(X) = 0$ which agrees with the fact that $\widetilde{H}_0(S(X)) = 0$ since $S(X)$ is necessarily connected).

Fall 2020-7. (a) Describe a cell decomposition for $X = \mathbb{RP}^4$ such that its 2-skeleton $X^{(2)} = \mathbb{RP}^2$. (This means that X is obtained from $X^{(2)}$ by attaching only 3-and 4-dimensional cells.) Include a careful description of the attaching maps.

(b) Use your cell decomposition to compute $H_k(X; \mathbb{Z})$ and $H_k(X, X^{(2)}; \mathbb{Z})$ for all $k \geqslant 0$.

Hint: One cell in each dimension with double cover for attaching maps. $H_k(\mathbb{RP}^n) = \mathbb{Z}$ if $k = n$ and n is odd, $\mathbb{Z}/2\mathbb{Z}$ if $k \leqslant n$ and k is odd and 0 otherwise.

Referenced in: Fall 2015-10, Spring 2012-6, Spring 2008-9.

(a) We can in fact construct a CW complex for $X = \mathbb{RP}^n$ which has $X^{(k)} = \mathbb{RP}^k$ for all $0 \leqslant k \leqslant n$. We need exactly one k-cell (i.e., D^k) for each $0 \leqslant k \leqslant n$. Then, the attaching maps are given by

$$\phi_k : S^{k-1} \to X^{(k-1)} = \mathbb{RP}^{k-1}$$

where ϕ_k is the standard double cover of $\mathbb{RP}^{k-1}$, namely $x, -x \mapsto [x]$ for all $x \in S^{k-1}$.

(b) Since there is one k-cell in each dimension k, our chain complex is

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to 0.$$

To compute the differential maps, we compute the degree of the boundary maps which are the compositions

$$S^{k-1} \xrightarrow{\phi_k} \mathbb{RP}^{k-1} \xrightarrow{q_k} \mathbb{RP}^{k-1}/\mathbb{RP}^{k-2} \cong S^{k-1}$$

where q_k is the canonical quotient map. We can see that $q_k \circ \phi_k$ restricts to a homeomorphism from each of the two components of $S^{k-1} - S^{k-2}$ to $\mathbb{RP}^{k-1} - \mathbb{RP}^{k-2}$. One of these homeomorphisms is given by precomposition with the antipodal map of S^{k-1} which has degree $(-1)^k$ and the other is given by the identity with degree 1. Putting this together gives

$$\deg(q_k \circ \phi_k) = \deg(\text{id}) + \deg(\text{antipodal}) = 1 + (-1)^k = \begin{cases} 0 & k \text{ odd}, \\ 2 & k \text{ even}. \end{cases}$$

This means we have

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0.$$

Hence, we have

$$H_k(X) = \begin{cases} \mathbb{Z} & k = 0, \\ \mathbb{Z}/2\mathbb{Z} & k = 1 \text{ or } 3, \\ 0 & \text{otherwise.} \end{cases}$$

To compute $H_k(X, X^{(2)}; \mathbb{Z})$, we have:

$$0 \to \frac{C_4(X)}{C_4(X^{(2)})} \to \frac{C_3(X)}{C_3(X^{(2)})} \to \frac{C_2(X)}{C_2(X^{(2)})} \to \frac{C_1(X)}{C_1(X^{(2)})} \to \frac{C_0(X)}{C_0(X^{(2)})}$$

but we chose $X^{(2)}$ so that

$$C_k(X^{(2)}) = \begin{cases} C_k(X) & 0 \leqslant k \leqslant 2, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, we have the complex

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to 0 \to 0 \to 0 \to 0$$

giving us:

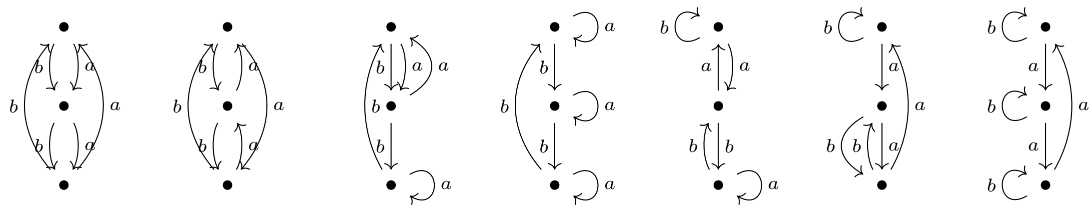
$$H_k(X, X^{(2)}; \mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & k = 3 \\ 0 & \text{otherwise.} \end{cases}$$

Fall 2020-8. List all the 3-sheet connected covering spaces of $S^1 \vee S^1$. Which ones in the list are not normal?

Hint: Graphs with 3 vertices of degree 4 each. Consider the edges a and b. Seven total, four normal.

Referenced in: Fall 2013-7.

Note that a covering space of a CW complex can be given a CW complex structure by lifting the characteristic maps to the covering space. Hence, a 3-sheet connected covering of $S^1 \vee S^1$ corresponds to a connected graph that has three vertices each of degree four (since $S^1 \vee S^1$ has one vertex of degree four). Further, if we give the standard labeling to $S^1 \vee S^1$, i.e., label one of the edges a and the other b, then each vertex of our covering space must have an incoming and outgoing a and b edge as well. We get the following options basically via systematic trial and error.



Of these, the first, second, fourth, and seventh are normal since the group action is transitive for these four and not for the other three.

Fall 2020-9. Let Σ_5 be a compact oriented surface of genus 5 without boundary. Does there exist an immersion $f : T^2 \to \Sigma_5$? Justify your answer.

Hint: Immersion $\implies$ local diffeomorphism $\implies$ covering map (since T^2 is compact and Σ_5 is connected) $\implies \chi(T^2) = k\chi(\Sigma_5)$ which is impossible.

By part (a) of Spring 2018-1, we know that such an f is a covering map since T^2 is compact and T^2 and Σ_5 are connected manifolds of the same dimension (2). Now (by Spring 2023-6) we know that a k-sheeted cover gives rise to the Euler characteristic formula: $\chi(T^2) = k\chi(\Sigma_5)$ and we know the Euler characteristic of a genus 5 surface is $2 - 2 \cdot 5 = -8$ so $\chi(T^2) = -8k$ for some $k \in \mathbb{N}_+$, a contradiction since $\chi(T^2) = 2 - 2 \cdot 1 = 0$.

Fall 2020-10. Show that the Euler characteristic of the special linear group $SL(n, \mathbb{R})$ with $n > 1$ is zero. Here for a topological space X its Euler characteristic is

$$\chi(X) := \sum_i (-1)^i \mathrm{rank}(H_i(X)),$$

assuming that $\sum_i \mathrm{rank}(H_i(X)) < \infty$.

Hint: Homotopy equivalent to $SO(n)$ which is parallelizable so has nowhere vanishing vector field. Also, compact so has $\chi = 0$ by Poincaré-Hopf. Use $A \mapsto \frac{(1-t)A+tU}{\det((1-t)A+tU)}$ for $A = UP$ polar decomposition.

Referenced in: Fall 2015-1, Fall 2010-3.

First, we claim that $SL_n(\mathbb{R})$ is homotopy equivalent to $SO(n)$. Note that we have a well-defined smooth map $r : SL_n(\mathbb{R}) \to SO(n)$ given by $A = UP \mapsto U$ where UP is the polar decomposition of A into U an orthogonal matrix and P a positive definite matrix. Let $i : SO(n) \hookrightarrow SL_n(\mathbb{R})$ be the inclusion. We know that the polar decomposition is unique so $r \circ i = \mathrm{id}$. We want to show that $i \circ r$ is homotopy equivalent to id. For $t \in [0, 1]$, consider

$$H_t : SL_n(\mathbb{R}) \to SL_n(\mathbb{R}), \quad A \mapsto \frac{(1-t)A + tU}{\det((1-t)A + tU)},$$

where $A = UP$ is the polar decomposition. Note that $\det((1-t)A+tU) \neq 0$. This is because $(1-t)A+tU = U((1-t)P+tI)$ and P, I are positive definite so the convex combination $(1-t)P+tI$ is also positive definite and also $\det(U) \neq 0$. Now, $H_0 = \mathrm{id}$ while $H_1 = i \circ r$, proving the claim.

Now we know that $SO(n)$ and $SL_n(\mathbb{R})$ have the same Euler characteristic (clear from the definition of χ in terms of homology only). But $SO(n)$ is a compact lie group. Thus, it is parallelizable (say by the second part of Fall 2022-2) so admits a nowhere vanishing vector field which implies by the Poincaré-Hopf index theorem that $\chi(SO(n)) = 0$ so $\chi(SL_n(\mathbb{R})) = 0$ as desired.

Spring 2020

Spring 2020-1. Suppose $f : M \to N$ is a smooth map between smooth manifolds, and is smoothly homotopic to a locally constant map. Prove $f^*\omega$ is exact for any closed differential k-form ω on N (with $k > 0$).

Hint: $f \sim g$. g is constant on the path components. So $g|_{M_i}^* = 0 \implies g^* = 0 \implies f^* = 0$ on the level of cohomology.

Let $g : M \to N$ be the locally constant function that f is homotopic to. We claim that g is constant on each path component of M. Let $a, b \in M$ be points in the same path component and consider a path $\gamma : [0, 1] \to M$ from a to b. For each $x \in \gamma([0, 1])$, let U_x be a neighborhood on which g is constant. Since $\gamma([0, 1])$ is compact, we can cover $\gamma([0, 1])$ with a finite number of these U_x's, say $U_{x_1}, \ldots, U_{x_n}$. It is easy to see inductively that g is then constant on the union of these U_{x_i}'s so in particular $g(a) = g(b)$ as claimed.

Now, let M_i be a path component of M. Then $g|_{M_i}$ is constant so $g|_{M_i}^* = 0$. But g^* is the sum of the individual $g|_{M_i}^*$ maps so $g^* = 0$. We know that cohomology is homotopy-invariant (say by Fall 2015-2) which implies that $f^*\omega = g^*\omega = 0$ since ω is closed. Thus, $f^*\omega = 0$ on the level of cohomology so is exact.

Spring 2020-2. Let M be a smooth 4-dimensional manifold. A symplectic form is a closed 2-form ω on M such that $\omega \wedge \omega$ is a nowhere vanishing 4-form.

(a) Construct a symplectic form on $\mathbb{R}^4$.
(b) Show that there are no symplectic forms on the unit sphere S^4.

Hint: $dx_1 \wedge dx_2 + dx_3 \wedge dx_4$. $H^2(S^4) = 0$ so closed implies exact, $\omega \wedge \omega$ is also exact so $\int_{S^4} \omega \wedge \omega = 0$, implying $\omega \wedge \omega$ is not nowhere vanishing.

(a) Let $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$. Then $d\omega = 0$ is clear and we have

$$\omega \wedge \omega = (dx_1 \wedge dx_2 + dx_3 \wedge dx_4) \wedge (dx_1 \wedge dx_2 + dx_3 \wedge dx_4) = dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 + dx_3 \wedge dx_4 \wedge dx_1 \wedge dx_2 = 2dV,$$

which is a nowhere vanishing 4-form.

(b) Since $H^2(S^4) = 0$, we know that any closed 2-form ω is exact. Then, $\omega \wedge \omega$ is also exact since if $\omega = d\nu$, then $d(\nu \wedge d\nu) = d\nu \wedge d\nu = \omega \wedge \omega$. So, by Stokes' theorem

$$\int_{S^4} \omega \wedge \omega = \int_{\partial S^4} \nu \wedge d\nu = 0$$

since S^4 has empty boundary. Thus, for any closed 2-form $\omega, \omega \wedge \omega$ is not nowhere vanishing so there are no symplectic forms on S^4.

Spring 2020-3. Consider the differential form $\omega = xdy - ydx - dz$ in $\mathbb{R}^3$ with coordinates (x, y, z). Prove that $f\omega$ is not closed for any nowhere zero smooth function $f : \mathbb{R}^3 \to \mathbb{R}$.

Hint: Just expand $d(f\omega) = df \wedge \omega + f d\omega$.

Referenced in: Fall 2017-4

Note that

$$d\omega = dx \wedge dy - dy \wedge dx = 2dx \wedge dy$$

and

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

so we have

$$d(f\omega) = df \wedge \omega + fd\omega = (2f + x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y})dx \wedge dy + (y\frac{\partial f}{\partial z} - \frac{\partial f}{\partial x})dx \wedge dz + (-x\frac{\partial f}{\partial z} - \frac{\partial f}{\partial y})dy \wedge dz.$$

So if $d(f\omega) = 0$, then $y\frac{\partial f}{\partial z} - \frac{\partial f}{\partial x} = 0, -x\frac{\partial f}{\partial z} - \frac{\partial f}{\partial y} = 0$, and $2f + x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 0$. Adding x times the first equation to y times the second yields $-x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y} = 0$ and adding this to the third gives $2f = 0$, contradicting the fact that f is nowhere zero. Thus, $d(f\omega) \neq 0$ so $f\omega$ is not closed.

Spring 2020-4. For any two smooth vector fields X, Y on a smooth manifold M, prove the formula

$$[L_X, \iota_Y] = \iota_{[X,Y]}$$

where L_X denotes the Lie derivative and ι_X is the contraction of vector field acting on differential forms.

Hint: Compute left hand side using $(\iota_Y\omega)(V_1, \ldots, V_{k-1}) = \omega(Y, V_1, \ldots, V_{k-1})$ and $(L_X\omega)(V_1, \ldots, V_k) = X(\omega(V_1, \ldots, V_k)) - \sum_{i=1}^k \omega(V_1, \ldots, V_{i-1}, [X, V_i], V_{i+1}, \ldots, V_k)$.

Referenced in: Fall 2023-2, Spring 2019-4, Fall 2015-3.

Let ω be a k-form for $k \geqslant 1$ and let $V_1, \ldots, V_k$ be smooth vector fields on M. By definition, we have

$$(\iota_Y \omega)(V_1, \ldots, V_{k-1}) = \omega(Y, V_1, \ldots, V_{k-1}),$$

$$(L_X \omega)(V_1, \ldots, V_k) = X(\omega(V_1, \ldots, V_k)) - \sum_{i=1}^{k} \omega(V_1, \ldots, V_{i-1}, [X, V_i], V_{i+1}, \ldots, V_k).$$

So, we have

$$\begin{aligned}(L_X \iota_Y \omega)(V_1, \ldots, V_{k-1}) &= (L_X(\iota_Y \omega))(V_1, \ldots, V_{k-1}) \\ &= X((\iota_Y \omega)(V_1, \ldots, V_{k-1})) - \sum_{i=1}^{k-1} (\iota_Y \omega)(V_1, \ldots, V_{i-1}, [X, V_i], V_{i+1}, \ldots, V_{k-1}) \\ &= X(\omega(Y, V_1, \ldots, V_{k-1})) - \sum_{i=1}^{k-1} \omega(Y, V_1, \ldots, V_{i-1}, [X, V_i], V_{i+1}, \ldots, V_{k-1}).\end{aligned}$$

On the other hand,

$$\begin{aligned}(\iota_Y L_X \omega)(V_1, \ldots, V_{k-1}) &= (\iota_Y(L_X \omega))(V_1, \ldots, V_{k-1}) \\ &= (L_X \omega)(Y, V_1, \ldots, V_{k-1}) \\ &= X(\omega(Y, V_1, \ldots, V_{k-1})) - \omega([X, Y], V_1, \ldots, V_{k-1}) \\ &\quad - \sum_{i=1}^{k-1} \omega(Y, V_1, \ldots, V_{i-1}, [X, V_i], V_{i+1}, \ldots, V_{k-1}) \\ &= (L_X \iota_Y \omega)(V_1, \ldots, V_{k-1}) - \omega([X, Y], V_1, \ldots, V_{k-1}) \\ &= (L_X \iota_Y \omega)(V_1, \ldots, V_{k-1}) - (\iota_{[X,Y]} \omega)(V_1, \ldots, V_{k-1}).\end{aligned}$$

Combining and rearranging gives

$$[L_X, \iota_Y]\omega = L_X \iota_Y \omega - \iota_Y L_X \omega = \iota_{[X,Y]} \omega,$$

showing the desired result. Note that for $k = 0$, the result is trivial.

Spring 2020-5. Show that the complex projective space $\mathbb{CP}^{2n}$ does not cover any manifold except itself.

Hint: $\pi_1(X)$ acts on $\mathbb{CP}^{2n}$ via deck transformations. Any map $g : \mathbb{CP}^{2n} \to \mathbb{CP}^{2n}$ has a fixed point by Lefschetz trace formula so they are all the identity.

Referenced in: Fall 2023-6, Fall 2015-8.

Suppose $p : \mathbb{CP}^{2n} \to X$ is a covering. Then, $\pi_1(X)$ acts on $\mathbb{CP}^{2n}$ by deck transformations. We claim that $\pi_1(X) = 1$ showing that the group of deck transformations of $\mathbb{CP}^{2n}$ is the trivial group so we must have $X = \mathbb{CP}^{2n}$. We know that deck transformations are determined uniquely by where they send a single point so it suffices to show that any $g \in \pi_1(X)$ has a fixed point as it then must be the identity. However, this is exactly part (a) of Spring 2023-2.

Spring 2020-6. Show that any continuous map from $S^2 \times S^2$ to $\mathbb{CP}^2$ must be of even degree.

Hint: Cohomology rings $H^*(\mathbb{CP}^2) = \mathbb{Z}[\alpha]/(\alpha^3), |\alpha| = 2$ and $H^*(S^2 \times S^2) = \mathbb{Z}[\beta, \gamma]/(\beta^2, \gamma^2), |\beta| = |\gamma| = 2$. Write $f^*(\alpha) = a\beta + b\gamma$ so $f^*(\alpha^2) = 2ab\beta\gamma$ so degree is $2ab$.

Referenced in: Fall 2015-7.

Consider $f^*: H^*(\mathbb{CP}^2) \to H^*(S^2 \times S^2)$. We know

$$H^*(\mathbb{CP}^2) = \frac{\mathbb{Z}[\alpha]}{(\alpha^3)}, \quad |\alpha| = 2.$$

Also,

$$H^n(S^2) = \begin{cases} \mathbb{Z} & n = 0, 2, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$H^*(S^2) = \frac{\mathbb{Z}[\beta]}{(\beta^2)}, \quad |\beta| = 2.$$

Since the cohomology groups are all finitely generated free $\mathbb{Z}$-modules, the Künneth formula tells us that:

$$H^*(S^2 \times S^2) = H^*(S^2) \otimes_{\mathbb{Z}} H^*(S^2) = \frac{\mathbb{Z}[\beta, \gamma]}{(\beta^2, \gamma^2)}, \quad |\beta| = |\gamma| = 2.$$

Now, suppose $f^*(\alpha) = a\beta + b\gamma$. Then, we can see that $f^*(\alpha^2) = (a\beta + b\gamma)^2 = 2ab\beta\gamma$. But by the cup product structure on $H^*(\mathbb{CP}^2)$ and $H^*(S^2 \times S^2)$, we know that α^2 is a generator for $H^4(\mathbb{CP}^2)$ and $\beta\gamma$ is a generator for $H^4(S^2 \times S^2)$. So the degree of f^* and thus of f is $2ab$ which is even.

Spring 2020-7. Prove that the relative homology groups $H_k(X, x)$ for different choices of basepoint x can be naturally identified with each other. That is, for every $k \geqslant 0$, every space X, and all pairs of points $x, y \in X$ (not necessarily in the same connected component), construct isomorphisms $\eta^X_{x,y}: H_k(X, x) \to H_k(X, y)$ satisfying

(a) $\eta^X_{x,x} = \text{Id}$ for all $x \in X$.
(b) $\eta^X_{y,z} \circ \eta^X_{x,y} = \eta^X_{x,z}$ for all $x, y, z \in X$.
(c) $f_* \circ \eta^X_{x,y} = \eta^Y_{f(x),f(y)} \circ f_*$ for all $x, y \in X$ and all continuous maps $f: X \to Y$.

(Hint: You might consider proving the case $k \geqslant 1$ first.)

Hint: Long exact sequence of the pair for $k > 0$. For $k = 0$, consider connected components.

As directed by the hint, we assume $k \geqslant 1$. We have a long exact sequence from the pair (X, x)

$$\cdots \to H_k(\{x\}) \to H_k(X) \to H_k(X, x) \to H_{k-1}(\{x\}) \to H_{k-1}(X) \to \cdots.$$

Since $k \geqslant 1$, this gives us an isomorphism $\phi^X_x : H_k(X) \to H_k(X, x)$. Given $x, y \in X$, we define $\eta^X_{x,y}$: $H_k(X, x) \to H_k(X, y)$ by $\phi^X_y \circ (\phi^X_x)^{-1}$. Clearly, we have $\eta^X_{x,x} = \text{Id}$ and $\eta^X_{y,z} \circ \eta^X_{x,y} = \eta^X_{x,z}$.

Let $f: X \to Y$ be a map. We then have $f_* \circ \phi^X_x = \phi^Y_{f(x)} \circ f_*$ and $f_* \circ (\phi^X_x)^{-1} = (\phi^Y_{f(x)})^{-1} \circ f_*$ by naturality of the exact sequence for the pair (X, x) in X. So

$$f_* \circ \eta^X_{x,y} = f_* \circ \phi^X_y \circ (\phi^X_x)^{-1} = \phi^Y_{f(y)} \circ f_* \circ (\phi^X_x)^{-1} = \phi^Y_{f(y)} \circ (\phi^Y_{f(x)})^{-1} \circ f_* = \eta^Y_{f(x),f(y)} \circ f_*.$$

For $k = 0$, this follows from the fact that $H_0(X)$ is just the free $\mathbb{Z}$-module with one generator for each connected component of X and a map $f_*: H_0(X) \to H_0(Y)$ just describes which connected components are mapped to each other so everything is trivial (but tedious when done correctly) to check.

Spring 2020-8. Assume the integral homology of a finite CW complex X is $\mathbb{Z}$ in grading $0, \mathbb{Z}/2$ in grading 2, $\mathbb{Z}/3$ in grading 3, and 0 in all other gradings. What is the cohomology of X with $\mathbb{Z}/6$ coefficients? Can you give an example of such a space X?

Hint: Universal coefficient theorem. $\mathrm{Ext}^1_R(R/(u), B) = B/uB$ and $\mathrm{Ext}^i_R(A, B) = 0$ for all i if A is projective (free). For the example, just attach cells via degree maps that seem to work (two 3-cells, one to making grading 2 correct and one to make grading 3 correct).

The universal coefficient theorem states that for R a principal ideal domain, G an R-module, and $n \geqslant 0$, we have

$$H^n(X; G) \cong \mathrm{Hom}_R(H_n(X; R), G) \oplus \mathrm{Ext}^1_R(H_{n-1}(X; R), G).$$

So in our case, we compute:

$$\begin{aligned}
H^0(X; \mathbb{Z}/6) &\cong \mathrm{Hom}_{\mathbb{Z}}(H_0(X; \mathbb{Z}), \mathbb{Z}/6) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(H_{-1}(X; \mathbb{Z}), \mathbb{Z}/6) \\
&= \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/6) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(0, \mathbb{Z}/6) = \mathbb{Z}/6 \oplus 0 = \mathbb{Z}/6, \\
H^1(X; \mathbb{Z}/6) &\cong \mathrm{Hom}_{\mathbb{Z}}(H_1(X; \mathbb{Z}), \mathbb{Z}/6) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(H_0(X; \mathbb{Z}), \mathbb{Z}/6) \\
&= \mathrm{Hom}_{\mathbb{Z}}(0, \mathbb{Z}/6) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/6) = 0 \oplus 0 = 0, \\
H^2(X; \mathbb{Z}/6) &\cong \mathrm{Hom}_{\mathbb{Z}}(H_2(X; \mathbb{Z}), \mathbb{Z}/6) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(H_1(X; \mathbb{Z}), \mathbb{Z}/6) \\
&= \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}/6) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(0, \mathbb{Z}/6) = \mathbb{Z}/2 \oplus 0 = \mathbb{Z}/2, \\
H^3(X; \mathbb{Z}/6) &\cong \mathrm{Hom}_{\mathbb{Z}}(H_3(X; \mathbb{Z}), \mathbb{Z}/6) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(H_2(X; \mathbb{Z}), \mathbb{Z}/6) \\
&= \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/3, \mathbb{Z}/6) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}/6) \\
&= \mathbb{Z}/3 \oplus (\mathbb{Z}/6)/(2\mathbb{Z}/6) = \mathbb{Z}/3 \oplus \mathbb{Z}/2 = \mathbb{Z}/6, \\
H^4(X; \mathbb{Z}/6) &\cong \mathrm{Hom}_{\mathbb{Z}}(H_4(X; \mathbb{Z}), \mathbb{Z}/6) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(H_3(X; \mathbb{Z}), \mathbb{Z}/6) \\
&= \mathrm{Hom}_{\mathbb{Z}}(0, \mathbb{Z}/6) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/3, \mathbb{Z}/6) = 0 \oplus (\mathbb{Z}/6)/(3\mathbb{Z}/6) = \mathbb{Z}/3,
\end{aligned}$$

and clearly $H^k(X; \mathbb{Z}/6) = 0$ for any other k. We build X as a CW complex. X has one 0-cell, one 2-cell, two 3-cells, and one 4-cell. The 2-cell is attached to the 0-cell in the only possible way. The 3-cells are then attached to the 2-cell via a degree 2 and a degree 0 map, respectively. Finally, the 4-cell is attached via a degree 3 map to the 3-cell that is attached via a degree 0 map.

Spring 2020-9. Consider the following group with $2n$ generators and 1 relation

$$G_n = \langle a_1, b_1, a_2, b_2, \ldots, a_n, b_n \mid a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1} \rangle.$$

For which pairs (m, n) does G_n contain a finite index subgroup isomorphic to G_m?

Hint: $m = 1 + k(n-1)$ for some $k \in \mathbb{N}$. Corresponds to coverings of an n-torus, which are exactly the m-tori such that $\chi(T_n) = 2 - 2n \mid 2 - 2m = \chi(T_m)$.

Referenced in: Fall 2023-8.

By inspection, note that $G_n = \pi_1(T_n)$ where T_n is the genus n torus. Thus, a subgroup of finite index k in G_n corresponds to a k-fold covering of T_n which must be a closed orientable 2-manifold. I.e., a k-fold covering of T_n is another torus T_m for some $m \geqslant 0$. Then, as in part (b) of Spring 2023-6, such a cover exists if and only if $m = 1 + k(n-1)$ for some $k \in \mathbb{N}$.

Spring 2020-10. Let D^2 be the unit disk in $\mathbb{C}$, and let $S^1 = \partial D^2$. Let $X = D^2 \times S^1 \times \{0, 1\}/\sim$ where

$$(x, y, 0) \sim (xy^5, y, 1)$$

for all $x, y \in S^1$. Compute the homology groups of X.

Hint: $H_*(X) = \mathbb{Z}_{(3)} \oplus \mathbb{Z}_{(2)} \oplus \mathbb{Z}_{(1)} \oplus \mathbb{Z}_{(0)}$. Mayer Vietoris for neighborhoods around $D^2 \times S^1 \times \{0\}, D^2 \times S^1 \times \{1\}$. Then $A \cap B \simeq \partial D^2 \times S^1 = S^1 \times S^1$.

Referenced in: Fall 2023-10.

Let A be a neighborhood of $D^2 \times S^1 \times \{0\}/\sim$ that deformation retracts onto $D^2 \times S^1 \times \{0\}/\sim \cong D^2 \times S^1$ and let B be a neighborhood of $D^2 \times S^1 \times \{1\}/\sim$ that deformation retracts onto $D^2 \times S^1 \times \{1\}/\sim \cong D^2 \times S^1$. Then, it can be seen that $A \cap B$ deformation retracts onto $\partial D^2 \times S^1 \cong S^1 \times S^1$. Thus, the Mayer-Vietoris sequence for A and B becomes

$$\cdots \to H_{n+1}(X) \to H_n(S^1 \times S^1) \to H_n(D^2 \times S^1) \oplus H_n(D^2 \times S^1) \to H_n(X) \to \cdots .$$

We know that D^2 is contractible so $H_*(D^2 \times S^1) \cong H_*(S^1) = \mathbb{Z}_{(1)} \oplus \mathbb{Z}_{(0)}$. Then, we can also compute $H_n(S^1 \times S^1)$ using Künneth formula,

$$H_*(S^1 \times S^1) = \mathbb{Z}_{(2)} \oplus \mathbb{Z}^2_{(1)} \oplus \mathbb{Z}_{(0)}.$$

So our long exact sequence becomes

$$0 \to H_3(X) \to \mathbb{Z} \to 0 \to H_2(X) \to \mathbb{Z}^2 \xrightarrow{f} \mathbb{Z} \oplus \mathbb{Z} \to H_1(X) \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to H_0(X) \to 0.$$

Hence, we have $H_3(X) \cong \mathbb{Z}$. Also it is clear that X is path-connected so $H_0(X) \cong \mathbb{Z}$. Hence, the map g is injective so $H_1(X) = \operatorname{coker}(f)$. By drawing pictures, we can see that f is defined by $(1,0) \mapsto (1,1), (0,1) \mapsto (0,0)$. Thus, $H_2(X) = \ker(f) \cong \mathbb{Z}$ and $H_1(X) = \operatorname{coker}(f) \cong \mathbb{Z}$. To summarize, we have

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0,1,2,3, \\ 0 & \text{otherwise.} \end{cases}$$

Fall 2019

Fall 2019-1. State the classical Divergence Theorem (also called Gauss' Divergence Theorem) for a compact 3-dimensional submanifold of $\mathbb{R}^3$ with smooth boundary. Derive it from Stokes' Theorem for differential forms.

Hint: $\iiint_M \operatorname{div}(X) dV = \iint_{\partial M} \langle X, n \rangle dS$ for n the outward pointing normal unit vector to ∂M and $dS = i^* \iota_n dV$ the induced volume form on ∂M. Recall $\operatorname{div}(X) dV = L_X(dV)$ and use Cartan's magic formula. Consider $Y = X - \langle X, n \rangle n$.

Referenced in: Fall 2018-5, Spring 2018-6, Spring 2010-9, Spring 2008-3.

Let $M \subset \mathbb{R}^3$ be an embedded submanifold of dimension 3 with smooth boundary. The divergence theorem states that given a vector field X on M,

$$\iiint_M \operatorname{div}(X) dV = \iint_{\partial M} \langle X, n \rangle dS,$$

where n is the outward pointing normal unit vector to ∂M and dS is the induced volume form on ∂M, coming from dV on M.

By definition, we have $\int_M \operatorname{div}(X) dV = \int_M L_X(dV)$. By Cartan's magic formula, $L_X(dV) = d\iota_X(dV) + \iota_X d(dV)$. But $d(dV) = 0$ since dV is a top form on M so

$$\int_M \operatorname{div}(X) dV = \int_M d(\iota_X dV) = \int_{\partial M} i^* \iota_X dV,$$

by Stokes' theorem, where $i : \partial M \hookrightarrow M$ is the inclusion. So now we want to show $i^* \iota_X dV = \langle X, n \rangle dS$.

To see this, note that $dS = i^* \iota_n dV$. Then, letting $Y = X - \langle X, n \rangle n$, we note that for any $p \in \partial M$, we have $Y_p \in T_p(\partial M)$ by definition of the normal vector. Now, for any vector fields X_1, X_2 tangent to ∂M,

$$i^* \iota_Y dV(X_1, X_2) = i^* dV(Y, X_1, X_2) = i^* 0 = 0$$

since Y, X_1, and X_2 are all vector fields in ∂M which is a 2-manifold so these vector fields are linearly dependent at each point p. Hence, $i^*\iota_Y dV = 0$. But, we also have

$$i^*\iota_Y dV = i^*\iota_X dV - \langle X, n\rangle i^*\iota_n dV$$

so indeed $\langle X, n\rangle dS = \langle X, n\rangle i^*\iota_n dV = i^*\iota_X dV$.

Fall 2019-2. Compute $H_*(\mathbb{RP}^{n+m}/\mathbb{RP}^n;\mathbb{Z})$ as a function of n and m. Here we are viewing $\mathbb{RP}^n \subset \mathbb{RP}^{n+m}$ induced from the inclusion

$$\mathbb{R}^{n+1} \subset \mathbb{R}^{n+m+1}, \quad (x_1, \ldots, x_{n+1}) \mapsto (x_1, \ldots, x_{n+1}, 0, \ldots, 0).$$

Hint: Give $\mathbb{RP}^{n+m}$ a CW structure with one cell in each dimension. Then, relative chain complex is just sequence of $\mathbb{Z}$'s followed by a sequence of 0's. Split into n even/odd cases. Funny business only for $i = n+1$ when n is odd.

Note that $(\mathbb{RP}^{n+m}, \mathbb{RP}^n)$ is a good pair so we have

$$\widetilde{H}_i(\mathbb{RP}^{n+m}/\mathbb{RP}^n;\mathbb{Z}) \cong H_i(\mathbb{R}^{n+m}, \mathbb{RP}^n;\mathbb{Z}).$$

Then, we computed this relative homology in Spring 2023-9.

Fall 2019-3. For which $n > 0$ does the real projective space $\mathbb{RP}^n$ admit a nowhere-vanishing vector field? If a nowhere vanishing vector field exists, give an explicit one.

Hint: If and only if n is odd. Standard nowhere zero vector field on S^n. Namely, $(x_1, x_2, \ldots, x_n, x_{n+1}) \mapsto (-x_2, x_1, \ldots, -x_{n+1}, x_n)$, factors through a vector field on $\mathbb{RP}^n$.

Referenced in: Fall 2012-2.

Suppose that X is a nowhere vanishing vector field on $\mathbb{RP}^n$. Since $\mathbb{RP}^n$ is compact, X has a global flow ϕ_t. For small $t > 0$, ϕ_t has no fixed points, so by the Lefschetz fixed point theorem, $L(\phi_t) = 0$. Then $\chi(\mathbb{RP}^n) = L(\phi_0) = L(\phi_t) = 0$. It is well-known that $\chi(\mathbb{RP}^n) = (1 + (-1)^n)/2$ so we require n to be odd for $\mathbb{RP}^n$ to admit a nowhere vanishing vector field.

We claim this is an if and only if, i.e., $\mathbb{RP}^n$ admits a nowhere vanishing vector if and only if n is odd. If n is odd, consider the vector field

$$Y : (x_1, x_2, \ldots, x_n, x_{n+1}) \mapsto (-x_2, x_1, \ldots, -x_{n+1}, x_n)$$

on $S^n \subset \mathbb{R}^{n+1}$. This is a vector field since

$$x \cdot Y(x) = (x_1, x_2, \ldots, x_n, x_{n+1}) \cdot (-x_2, x_1, \ldots, -x_{n+1}, x_n) = -x_1x_2 + x_2x_1 - \ldots - x_nx_{n+1} + x_{n+1}x_n = 0$$

and Y never vanishes since $0 \notin S^n$. This corresponds to a section $Y : S^n \to TS^n$ of the standard projection $TS^n \to S^n$. Now, consider the usual double cover $\pi : S^n \to \mathbb{RP}^n$ which induces $d\pi : TS^n \to T\mathbb{RP}^n$. We claim that $d\pi \circ Y : S^n \to T\mathbb{RP}^n$ factors through $\mathbb{RP}^n$, hence giving us a map $X : \mathbb{RP}^n \to T\mathbb{RP}^n$ which is nowhere 0 since Y is nowhere 0.

In order to factor through $\mathbb{RP}^n$, we require $(d\pi \circ Y)(v) = (d\pi \circ Y)(-v)$ for all $v \in S^n$ so that $(d\pi \circ Y)$ is well-defined on each antipodal equivalence pair. For this, note that

$$(d\pi \circ Y)(v) = d\pi(v, Y(v)) = ([v], [Y(v)])$$

and $[v] = [-v]$ in $\mathbb{RP}^n$, and $Y(-v) = -Y(v)$ so $[Y(v)] = [Y(-v)]$ in $T_v\mathbb{RP}^n$ so indeed $d\pi \circ Y$ factors through $\mathbb{RP}^n$ and we are done.

Fall 2019-4. Let $X = S^1 \times S^1$ and let Y be the quotient of $X \times [0,1]$ by the relation

$$((x,y),0) \sim ((y,x),1).$$

Compute $H_*(Y;\mathbb{Z})$.

Hint: $H_*(Y) = \mathbb{Z}_{(2)} \oplus \mathbb{Z}/2\mathbb{Z}_{(2)} \oplus \mathbb{Z}^2_{(1)} \oplus \mathbb{Z}_{(0)}$. Have to analyze $s_* : H_k(X) \to H_k(X)$ for $s : X \to X, (x,y) \mapsto (y,x)$ to get the specific maps in the Mayer Vietoris sequence.

Define

$$A = (X \times [0,0.4) \cup X \times (0.6,1])/\sim, \quad B = (X \times (0.1,0.8))/\sim.$$

Note that $A \cup B = Y$, both A and B deformation retract onto a copy of X, and $A \cap B$ deformation retracts onto two disjoint copies of X. We also know, by Künneth's formula, that $H_*(X) = H_*(S^1 \times S^1) = \mathbb{Z}_{(2)} \oplus \mathbb{Z}^2_{(1)} \oplus \mathbb{Z}_{(0)}$. I.e.,

$$H_*(A) \cong H_*(B) \cong H_*(X) = \mathbb{Z}_{(2)} \oplus \mathbb{Z}^2_{(1)} \oplus \mathbb{Z}_{(0)}, \quad H_*(A \cap B) \cong H_*(X) \oplus H_*(X) = \mathbb{Z}^2_{(2)} \oplus \mathbb{Z}^4_{(1)} \oplus \mathbb{Z}^2_{(0)}.$$

Hence, the long exact sequence coming from Mayer-Vietoris looks like

$$0 \to H_3(Y) \to \mathbb{Z}^2 \xrightarrow{f} \mathbb{Z}^2 \to H_2(Y) \to \mathbb{Z}^4 \xrightarrow{g} \mathbb{Z}^4 \to H_1(Y) \to \mathbb{Z}^2 \xrightarrow{h} \mathbb{Z}^2 \to H_0(Y) \to 0.$$

It is clear that Y is path connected so $H_0(Y) \cong \mathbb{Z}$. Then, note that the maps f, g, and h are induced by the inclusions $A \cap B \hookrightarrow A$ and $A \cap B \hookrightarrow B$. The inclusion $A \cap B \hookrightarrow A$ corresponds to the map $X \sqcup X \to X$ which is the identity on both copies of X while the inclusion $A \cap B \hookrightarrow B$ corresponds to the map $X \sqcup X \to X$ which is the identity on the first copy of X and is $(x,y) \mapsto (y,x)$ on the second copy.

Hence, the maps f and h are given by

$$(a,b) \mapsto (a+b, a+s_*(b)) \text{ where } s : X \to X \text{ is given by } (x,y) \mapsto (y,x)$$

while g is

$$((a,b),(c,d)) \mapsto ((a,b)+(c,d),(a,b)+s_*(c,d)).$$

Now, we seek to find $s_* : H_k(X) \to H_k(X)$ for $k = 0, 1$, and 2. By inspection, we see that for $k = 0, s_*$ is just the identity and for $k = 2, s_*$ is multiplication by -1 since the map s swaps the entries of X so has degree -1. Finally, for $k = 1$, we have $s_*(c,d) = (d,c)$. Hence, we can explicitly write

$$f(a,b) = (a+b, a-b), \quad h(a,b) = (a+b, a+b), \quad g(a,b,c,d) = (a+c, b+d, a+d, b+c).$$

Note that $\operatorname{im}(f) = \{(c, 2a-c) \mid a, c \in \mathbb{Z}\}$ so $\operatorname{coker}(f) = \mathbb{Z}^2/(\mathbb{Z} \oplus 2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. Hence,

$$\begin{aligned} H_3(Y) &= \ker(f) \cong 0, \\ H_2(Y) &= \operatorname{coker}(f) \oplus \ker(g) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}, \\ H_1(Y) &= \operatorname{coker}(g) \oplus \ker(h) \cong \mathbb{Z} \oplus \mathbb{Z} \cong \mathbb{Z}^2. \end{aligned}$$

To summarize, we have

$$H_k(Y) = \begin{cases} \mathbb{Z} & k = 0, \\ \mathbb{Z}^2 & k = 1, \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & k = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Fall 2019-5. A vector field X on a Lie group G is *left-invariant* if $(L_g)_* X = X$ for all $g \in G$, where $L_g : G \to G$, $g' \mapsto gg'$, is left multiplication by g. Show that if X, Y are left-invariant vector fields, then so is their Lie bracket $[X,Y]$ as vector fields. You must prove any facts about Lie brackets that you use.

Hint: Just compute. Recall $F_*X(f) = X(f \circ F) \circ F^{-1}$ where F_* is the pushforward of F.

Let $g \in G$ and let $f \in C^\infty(G;\mathbb{R})$. Then, we compute

$$\begin{aligned}(L_g)_*[X,Y](f) &= [X,Y](f \circ L_g) \circ L_{g^{-1}} \\ &= XY(f \circ L_g) \circ L_{g^{-1}} - YX(f \circ L_g) \circ L_{g^{-1}} \\ &= X((L_g)_*Y(f) \circ L_g) \circ L_{g^{-1}} - Y((L_g)_*X(f) \circ L_g) \circ L_{g^{-1}} \\ &= X(Yf \circ L_g) \circ L_{g^{-1}} - Y(Xf \circ L_g) \circ L_{g^{-1}} \\ &= (L_g)^*X(Yf) - (L_g)^*Y(Xf) \\ &= XY(f) - YX(f) \\ &= [X,Y](f),\end{aligned}$$

where the fourth and sixth equalities come from the fact that X and Y are left invariant. So indeed, $(L_g)_*[X,Y] = [X,Y]$ for any $g \in G$.

Fall 2019-6. Let $Z(P)$ be the zero set of a degree d homogeneous polynomial $P(z_0, z_1, z_2)$ in $\mathbb{CP}^2$ with respect to the homogeneous coordinates $[z_0 : z_1 : z_2]$. Assuming P has no repeated factors, give necessary and sufficient conditions on P for $Z(P)$ to be smooth.

Hint: Unsure. Regular value, preimage theorem.

Note that $Z(P) = P^{-1}(0)$. The polynomial P is certainly a smooth map so $Z(P)$ is a smooth manifold as long as 0 is a regular value of P by the preimage theorem. I.e., for any $z \in Z(P)$, we require dP_z to be surjective. Now, dP_z maps to a one-dimensional manifold so is surjective if and only if it is not identically zero. But $dP_z = 0$ precisely when z is a multiple root of P since dP is the differential of P.

Hence, $Z(P)$ is smooth if P does not have any multiple roots. (Speculation) In fact, this condition is also necessary. Suppose P had a multiple root $z = [z_0 : z_1 : z_2] \in \mathbb{CP}^2$, then the variety $Z(P)$ in $\mathbb{CP}^2$ has a kink/singularity at z which means it cannot be smooth.

Fall 2019-7. If $P(z_0, z_1, z_2) = z_0^d + z_1^d + z_2^d$ from the previous problem, then compute the Euler characteristic of $Z(P)$. You must show all of your work.

Hint: Unsure. $\chi(Z(P)) = 3d - d^2$.

First, note that $Z(P)$ is the union of d lines, which must intersect at $d(d-1)/2$ points in $\mathbb{C}^3$. Thus, this has Euler characteristic equal to $2d$ (for the lines) minus 2 times the number of points of intersection. Namely $2d - 2d(d-1)/2 = 3d - d^2$.

Fall 2019-8. Let $X = \{N, S, E, W\}$ with the topology given by

$$\{\varnothing, \{E\}, \{W\}, \{E, W\}, \{N, E, W\}, \{S, E, W\}, X\}.$$

For each n, find a path-connected degree n cover. Describe the universal cover.

Hint: X is the quotient of S^1 where we identify the northern hemisphere to a point and likewise for the southern. Think of a circle with $4n$ sections that go $N, E, S, W, N, E, S, W, N, \ldots$ around the circle.

Note that $X = S^1/\sim$ where $(x, y) \sim (x', y')$ when x and x' (are nonzero and) have the same sign. I.e., X is the space obtained by taking S^1 and identifying the points in the open western hemisphere and separately identifying the points in the open eastern hemisphere. Thus, an n-fold cover of X can be constructed by taking an n-fold cover of S^1 and quotienting in the same way.

Explicitly, for each n, take

$$\widetilde{X}_n = \{N_a, S_a, E_a, W_a\}_{a \in \mathbb{Z}/n\mathbb{Z}}$$

with a basis of open sets given by

$$\{\varnothing, \{E_a\}, \{W_a\}, \{E_a, S_a, W_a\}, \{W_a, N_{a+1}, E_{a+1}\}\}_{a \in \mathbb{Z}/n\mathbb{Z}}$$

(Note that the arithmetic $a+1$ is done mod n). Then, we define the covering map

$$p_n : \widetilde{X}_n \to X \text{ by } E_a \mapsto E, W_a \mapsto W, N_a \mapsto N, S_a \mapsto S$$

for each $a \in \mathbb{Z}/n\mathbb{Z}$. We now show this is actually an n-fold cover. For $W \in X$, we have the open neighborhood $\{W\} \subset X$ onto which p_n maps the n disjoint open sets $\{W_a\}_{a \in \mathbb{Z}/n\mathbb{Z}}$ homeomorphically. Similarly for $E \in X$. For $N \in X$, we have the open neighborhood $\{N, E, W\} \subset X$ which pulls back to $p_n^{-1}(\{N, E, W\}) = \bigcup_{a \in \mathbb{Z}/n\mathbb{Z}} \{W_a, N_{a+1}, E_{a+1}\}$ where each $\{W_a, N_{a+1}, E_{a+1}\}$ are clearly disjoint and map homeomorphically to $\{N, E, W\}$ by construction. Similarly for $S \in X$ so indeed p_n is a covering map.

The universal cover is constructed in exactly the same way but instead of $a \in \mathbb{Z}/n\mathbb{Z}$, we just have $a \in \mathbb{Z}$ and everything works out the same way.

Fall 2019-9. (a) If

$$0 \to A \to B \to C \to 0$$

is a short exact sequence of chain complexes, show how to get the boundary map in the associated long exact sequence.

(b) Compute the boundary map when the short exact sequence is the result of tensoring the chain complex

$$\cdots \to 0 \to \mathbb{Z} \xrightarrow{5} \mathbb{Z} \to 0 \to \cdots$$

with the short exact sequence

$$0 \to \mathbb{Z}/5 \xrightarrow{5} \mathbb{Z}/25 \to \mathbb{Z}/5 \to 0.$$

Hint: Snake lemma. Follow proof of snake lemma to get identity map on $\mathbb{Z}/5$.

(a) This is exactly the snake lemma. See part (b) of Fall 2022-6.

(b) We have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}/5 & \xrightarrow{\cdot 5} & \mathbb{Z}/25 & \longrightarrow & \mathbb{Z}/5 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \cdot 5} & & \downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & \mathbb{Z}/5 & \xrightarrow[\cdot 5]{} & \mathbb{Z}/25 & \longrightarrow & \mathbb{Z}/5 & \longrightarrow & 0 \end{array}$$

Since the right vertical map is 0, the homology group on the top right is $\mathbb{Z}/5$ which is generated by 1 so we only care where 1 gets sent. Then, $1 \in \mathbb{Z}/25$ gets sent to $1 \in \mathbb{Z}/5$ along the top row. And this gets sent to $5 \in \mathbb{Z}/25$ in the bottom middle which is the image of $1 \in \mathbb{Z}/5$ in the bottom left. Hence, the boundary map is just the identity on $\mathbb{Z}/5$.

Fall 2019-10. Let X be a path-connected, locally path-connected, semi-locally simply-connected space and let $\widetilde{X} \to X$ be the universal cover.

(a) Given $x_0 \in X$, explain how $\pi_1(X, x_0)$ can be viewed as the set of deck transformations of $\widetilde{X}$.
(b) Show that any map $\sigma : \Delta^n \to X$ lifts to a map $\widetilde{\sigma} : \Delta^n \to \widetilde{X}$, where Δ^n is the standard n-simplex.
(c) Show that the action of $\pi_1(X, x_0)$ on the set of maps $\widetilde{\sigma} : \Delta^n \to \widetilde{X}$ is free (i.e., if $g \in \pi_1(X, x_0)$ and $g \circ \widetilde{\sigma} = \widetilde{\sigma}$, then $g = \mathrm{id}$).
(d) Show that if $\widetilde{\sigma}_1, \widetilde{\sigma}_2 : \Delta^n \to \widetilde{X}$ are two lifts of σ, then there exists $g \in \pi_1(X, x_0)$ such that $g \circ \widetilde{\sigma}_1 = \widetilde{\sigma}_2$.

Hint: Use unique path lifting property for all parts (for (a) twice). Deck transformations act transitively on fibers.

Referenced in: Spring 2018-10.

(a) Pick a pre-image $\widetilde{x}_0 \in \widetilde{X}$ of $x \in X$. Then, call the cover $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$. For $\gamma \in \pi_1(X, x_0)$, by the unique lifting criterion, there exists a unique lift $\widetilde{\gamma} : (I, 0) \to (\widetilde{X}, \widetilde{x}_0)$ of $\gamma : (I, 0) \to (X, x_0)$. By the unique lifting criterion again, there exists a unique lift $f : (\widetilde{X}, \widetilde{x}_0) \to (\widetilde{X}, \widetilde{\gamma}(1))$ of $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$. This is a deck transformation. Conversely, given any deck transformation $f : \widetilde{X} \to \widetilde{X}$, take any path $\gamma : [0, 1] \to \widetilde{X}$ with $\gamma(0) = \widetilde{x}_0$ and $\gamma(1) = f(\widetilde{x}_0)$, and then the composition $p \circ \gamma$ defines an element in $\pi_1(X, x_0)$.

(b) Since Δ^n is contractible, $\pi_1(\Delta^n, *) = 1$. Hence $\sigma_*(\pi_1(\Delta^n, *)) = 0 \subset p_*(\pi_(\widetilde{X}, \widetilde{x}_0))$ so σ can be lifted to $\widetilde{\sigma} : \Delta^n \to \widetilde{X}$ by the lifting property for covering maps.

(c) Let $y \in \Delta^n$. Suppose $g \in \pi_1(X, x_0)$ and $g \circ \widetilde{\sigma} = \widetilde{\sigma}$. Then g, id are both lifts of $p : \widetilde{X} \to X$ that agree on $\widetilde{\sigma}(y)$. By the unique lifting property, $g = \mathrm{id}$.

(d) It suffices to show that for any $y \in \Delta^n$, there is some $g \in \pi_1(X, x_0)$ such that $g \circ \widetilde{\sigma}_1(y) = \widetilde{\sigma}_2(y)$ since if two lifts agree at one point, they must be the same lift since Δ^n is clearly connected. However, this is easy to see since both $\widetilde{\sigma}_1(y)$ and $\widetilde{\sigma}_2(y)$ are lifts of the point $\sigma(y) \in X$ and we know that the group of deck transformations $(\pi_1(X, x_0))$ acts transitively on the fibers of the covering space.

Spring 2019

Spring 2019-1. Let M be a smooth manifold. Show that there exists a smooth proper map $\rho : M \to \mathbb{R}$.

Hint: Use σ-compactness to find an open cover by precompact sets. Subordinate partition of unity, $f(x) = \sum_{i \in \mathbb{N}} i\lambda_i(x)$.

We know that any smooth manifold M is σ-compact. So, we may find a countable collection of precompact open sets $(U_i)_{i \in \mathbb{N}}$ that cover M. Hence, we can find a partition of unity $\{\lambda_i\}_{i \in \mathbb{N}}$ subordinate to this cover. Define

$$f(x) = \sum_{i=1}^{\infty} i\lambda_i(x).$$

Note that f is always positive since $\sum_{i=1}^{\infty} \lambda_i = 1$ and $0 \leqslant \lambda_i(x) \leqslant 1$ for all i. We show that f is proper by showing that $f^{-1}([-N, N])$ is compact for any $N \in \mathbb{R}^+$. Then, because any compact $K \subset \mathbb{R}$ is closed and is contained in some $[-N, N]$, we would have that $f^{-1}(K)$ is closed and contained in the compact set $f^{-1}([-N, N])$ so is compact since M is Hausdorff.

To show the claim, suppose $x \in f^{-1}([-N, N])$ so $f(x) \leqslant N$. Then, $x \in \bigcup_{i=1}^{N} U_i$. This is because, if $x \notin \bigcup_{i=1}^{N} U_i$, then $\lambda_i(x) = 0$ for all $i \leqslant N$, implying that

$$f(x) = \sum_{i=1}^{\infty} i\lambda_i(x) = \sum_{i=N+1}^{\infty} i\lambda_i(x) \geqslant (N+1) \sum_{i=N+1}^{\infty} \lambda_i(x) = (N+1) \sum_{i=1}^{\infty} \lambda_i(x) = N + 1 > N.$$

Hence, we have

$$f^{-1}([-N, N]) \subset \bigcup_{i=1}^{N} U_i \subset \bigcup_{i=1}^{N} \overline{U_i}$$

which is compact as it is the finite union of compact sets. Thus, $f^{-1}([-N, N])$ is compact as a closed subset of a compact set.

Spring 2019-2. A smooth manifold Y of dimension n is called *parallelizable* if there exists n vector fields $v^1, \dots, v^n$ on Y such that at every $y \in Y$, the tangent vectors $v_y^1, \dots, v_y^n$ are linearly independent.

Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be a smooth function. Suppose that $0 \in \mathbb{R}$ is a regular value, and let M be the smooth manifold $f^{-1}(\{0\})$. Show that $M \times S^1$ is parallelizable.

Hint: Show tangent bundle is trivial. Normal bundle NM is trivial one-dimensional and same with TS^1. Tangent bundle acrobatics.

Since M is an embedded submanifold of $\mathbb{R}^{n+1}$, we know that for any $p \in M$, we have

$$T_p(\mathbb{R}^{n+1}) = T_p(M) \oplus N_p(M)$$

where $N_p(M)$ is the normal bundle to M at p. Also

$$T(M \times S^1) = \pi_1^* TM \oplus \pi_2^* TS^1$$

where $\pi_1 : M \times S^1 \to M$ and $\pi_2 : M \times S^1 \to S^1$ are the standard projections. By the preimage theorem, we see that $T_p(M)$ has dimension n so $N_p(M)$ has dimension 1. In fact, letting

$$X = \sum_{i=1}^{n+1} \frac{\partial f}{\partial x_i} dx_i,$$

we see that X_p spans $N_p(M)$ for any $p \in M$. Moreover, X cannot be zero on M since every point $m \in M$ is a regular point so the differential $df = \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i}$ is surjective and so nonzero at m. Thus $N(M) = \text{Span}(X)$ is a trivial one-dimensional bundle.

Also, it is well-known that S^1 has trivial (one-dimensional) tangent bundle since it has a smooth nowhere zero vector field $(x, y) \mapsto (-y, x)$. Hence, we have

$$\begin{aligned}
T(M \times S^1) &= \pi_1^* TM \oplus \pi_2^* TS^1 \\
&\cong \pi_1^* TM \oplus \pi_2^* (S^1 \times \mathbb{R}) \\
&\cong \pi_1^* TM \oplus (M \times S^1 \times \mathbb{R}) \\
&\cong \pi_1^* (TM \oplus NM) \\
&\cong \pi_1^* (M \times \mathbb{R}^{n+1}) \\
&\cong M \times S^1 \times \mathbb{R}^{n+1},
\end{aligned}$$

which is trivial. Hence, $M \times S^1$ is parallelizable since its tangent bundle is (globally) trivial.

Spring 2019-3. Show that the antipodal map $A : S^n \to S^n, A(x) = -x$ is homotopic to the identity if and only if n is odd. (Feel free to use Lefschetz theory if you like.)

Hint: Degree is homotopy invariant, $\deg(A) = (-1)^{n+1}$. Block diagonal matrix with blocks rotation by πt for $0 \leqslant t \leqslant 1$.

Referenced in: Spring 2014-3.

Note that the antipodal map is the composition of $n+1$ reflections so has degree $(-1)^{n+1}$. If n is even, then $\deg(A) = -1$. However, we know that the degree of a map is homotopy invariant and that $\deg(\text{id}) = 1$, so A cannot be homotopic to the identity if n is even.

On the other hand, if n is odd, this means $S^n \subset \mathbb{R}^{n+1}$ with $n+1$ even so let H_t be the block diagonal matrix with 2×2 blocks

$$\begin{pmatrix} \cos(\pi t) & \sin(\pi t) \\ -\sin(\pi t) & \cos(\pi t) \end{pmatrix}$$

for $0 \leqslant t \leqslant 1$. Then, consider the homotopy $H : S^n \times [0, 1] \to S^n$ given by $H(x, t) = H_t(x)$. It is easy to see that $H_1 = -I_{n+1}$ and $H_0 = I_{n+1}$. Now, $-I_{n+1}$ indeed represents the antipodal map so H is a homotopy from the identity map to the antipodal map on S^n.

Spring 2019-4. For a vector field X on a smooth manifold M, we denote by $L_X : \Omega^k(M) \to \Omega^k(M)$ the Lie derivative of X acting on k-forms.

Prove that, for any vector fields X and Y, we have the Lie bracket identity

$$[L_X, L_Y] = L_{[X,Y]}.$$

Hint: Show/use $[L_X, \iota_Y] = \iota_{[X,Y]}$, Cartan's magic formula, and $L_x d = dL_x$.

By Spring 2020-4, we know that $[L_X, \iota_Y] = \iota_{[X,Y]}$. Also, by Cartan's magic formula, we have $L_X = d\iota_X + \iota_X d, L_Y = d\iota_Y + \iota_Y d$. Suppose ω is a differential form and consider

$$dL_X\omega = dd\iota_X\omega + d\iota_X d\omega = d\iota_X d\omega = d\iota_X d\omega + \iota_X dd\omega = L_X d\omega.$$

Hence, we have

$$\begin{aligned}
[L_X, L_Y] &= [L_X, d\iota_Y + \iota_Y d] \\
&= [L_X, d\iota_Y] + [L_x, \iota_Y d] \\
&= L_X d\iota_Y - d\iota_Y L_X + L_X \iota_Y d - \iota_Y dL_X \\
&= dL_X \iota_Y - d\iota_Y L_X + L_X \iota_Y d - \iota_Y L_X d \\
&= d[L_X, \iota_Y] + [L_X, \iota_Y]d \\
&= d\iota_{[X,Y]} + \iota_{[X,Y]}d \\
&= L_{[X,Y]}
\end{aligned}$$

Spring 2019-5. Show that a closed 1-form ω on a manifold M is exact if and only if $\int_{S^1} f^*\omega = 0$ for every smooth map $f : S^1 \to M$.

Hint: Stokes'. Define $g(x) = \int_{\gamma_x} \omega$ where γ_x is path from x_0 to x. Well-defined since integrating any loop gives 0 by assumption. Then $dg = \omega$.

Referenced in: Fall 2013-4, Spring 2011-4, Spring 2009-1.

If ω is an exact 1-form on M, then there is some $\eta \in \Omega^0(M)$ such that $d\eta = \omega$. Then,

$$\int_{S^1} f^*\omega = \int_{S^1} f^*(d\eta) = \int_{S^1} d(f^*\eta) = \int_{\partial S^1} f^*\eta = 0$$

by Stokes' and since S^1 has no boundary. Conversely, suppose $\int_{S^1} f^*\omega = 0$ for every $f : S^1 \to M$. Let $x_0 \in M$ and let $N \subset M$ be the path component of M containing x_0. Define

$$g_N : N \to \mathbb{R}, \quad x \mapsto \int_{\gamma_x} \omega,$$

where γ_x is a path from x_0 to x. Note that if γ_x and γ'_x are two paths from x_0 to x, let $f = \gamma_x \cdot \widetilde{\gamma'_x}$ be the loop formed by concatenating γ_x and the reverse of γ'_x. Without loss of generality we can take γ_x and γ'_x to be smooth (by homotoping them) so that $f : S^1 \to M$ is also smooth. Hence

$$0 = \int_{S^1} f^*\omega = \int_{f_*} \omega = \int_{\gamma_x} \omega - \int_{\gamma'_x} \omega,$$

implying that $\int_{\gamma_x} \omega = \int_{\gamma'_x}$ so g_N is well-defined. Now, do this on each path component of M and add to form $g : M \to \mathbb{R}$. We claim that $dg = \omega$. By the fundamental theorem of calculus, we have $dg_p = \frac{d}{dt}\big|_{t=1} \int_{\gamma_p} \omega = \omega_p$ as desired so ω is exact.

Spring 2019-6. Let $f : X \to Y$ be a smooth, finite covering map between smooth manifolds. Show that the induced map on de Rham cohomology

$$f^* : H^k_{dR}(Y;\mathbb{R}) \to H^k_{dR}(X;\mathbb{R})$$

is injective.

Hint: $U \subset Y$ evenly covered by $f^{-1}(U) = \bigsqcup_{i=1}^k U_i$ with local inverses $\phi_i : U \to U_i$ to f. Then, define $g(\omega)|_U = \frac{1}{k}\sum_{i=1}^k \phi_i^*\omega$ which is well-defined and satisfies $g \circ f^* = \text{id}$.

Referenced in: Fall 2018-7, Fall 2014-3, Spring 2012-9, Spring 2010-5.

We construct a retraction $g : H^k_{dR}(X) \to H^k_{dR}(Y)$ of f^* (i.e., $g \circ f^* = \text{id}$). Then f^* must be injective. Let $U \subset Y$ be an open set that is evenly covered, so $f^{-1}(U) = \bigsqcup_{i=1}^k U_i$ where $\phi_i : U \to U_i$ are (local) inverses to $f|_{U_i}$ for each i. For any closed form $\omega \in \Omega^k(X)$, we define

$$g(\omega)|_U = \frac{1}{k}\sum_{i=1}^k \phi_i^*\omega|_{U_i}.$$

Note that this defines $g(\omega)$ on all of M since the intersection of two evenly covered open sets is itself evenly covered and the local inverses agree on this intersection since f is the same as itself on the intersection. To see that this is well-defined, suppose that ω, ω' are the same up to cohomology. Then, $\omega = \omega' + d\psi$ for some $(n-1)$-form ψ. Then, on an evenly covered neighborhood $U \subset Y$, we have

$$\begin{aligned} g(\omega')|_U = \frac{1}{k}\sum_{i=1}^k \phi_i^*(\omega + d\psi)|_{U_i} &= \frac{1}{k}\sum_{i=1}^k \phi_i^*\omega|_{U_i} + \frac{1}{k}\sum_{i=1}^k \phi_i^* d\psi|_{U_i} = \frac{1}{k}\sum_{i=1}^k \phi_i^*\omega|_{U_i} + \frac{1}{k}\sum_{i=1}^k d(\phi_i^*\psi|_{U_i}) \\ &= \frac{1}{k}\sum_{i=1}^k \phi_i^*\omega|_{U_i} + d\left(\frac{1}{k}\sum_{i=1}^k \phi_i^*\psi|_{U_i}\right) = g(\omega)|_U + d\eta_U, \end{aligned}$$

for $\eta_U = \frac{1}{k}\sum_{i=1}^k \phi_i^*\psi|_{U_i}$ an $(n-1)$-form on U. These η_U's can similarly be glued together (since they agree on the overlap of evenly covered opens) to define η on all of Y such that $g(\omega') = g(\omega) + d\eta$. Finally, to see that $g \circ f^* = \text{id}$, we observe that locally we have

$$g(f^*\omega)|_U = \frac{1}{k}\sum_{i=1}^k \phi_i^*(f^*\omega)|_{U_i} = \frac{1}{k}\sum_{i=1}^k ((f\circ\phi_i)^*\omega)|_U = \frac{1}{k}\sum_{i=1}^k (\text{id}^*\omega)|_U = \frac{1}{k}\sum_{i=1}^k \omega|_U = \omega|_U.$$

Spring 2019-7. Let $X = [0,1]$ and $A = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{Z}, n \geqslant 1\}$. Show that $H_1(X, A)$ is not isomorphic to $H_1(X/A)$.

Hint: Long exact sequence for the pair shows $H_1(X, A)$ is countably generated. But X/A is the Hawaiian earring for which we define a surjection $\pi_1(X/A) \to \prod_{n=1}^\infty \mathbb{Z}$.

In degree one, the long exact sequence for the pair (X, A) is:

$$\cdots \to H_1(A) \to H_1(X) \to H_1(X,A) \xrightarrow{f} H_0(A) \to \cdots.$$

X is contractible so $H_1(X) = 0$, implying that f is an injection. Since A has countably many connected components, $H_0(A)$ is countably generated so $H_1(X, A)$ is also countably generated. On the other hand, we claim that $\pi_1(X/A)$ is not countably generated so $H_1(X/A)$, as the abelianization of $\pi_1(X/A)$, is also not countably generated, showing $H_1(X, A) \ncong H_1(X/A)$.

To show this, we exhibit a surjection

$$g : \pi_1(X/A) \to \prod_{n=1}^{\infty} \mathbb{Z}.$$

In X/A, we note that $\left[\frac{1}{n+1}, \frac{1}{n}\right]$ is a loop for each $n \in \mathbb{Z}, n \geqslant 1$. Moreover, for $n \neq m$, these loops are clearly not homotopic (in fact X/A is the Hawaiian earring). So we let g send $\left[\frac{1}{n+1}, \frac{1}{n}\right]$ to the element of $\prod_{i=1}^{\infty} \mathbb{Z}$ with 0's everywhere except a 1 in the nth entry.

Then, for any $a = (a_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} \mathbb{Z}$, we can construct the loop γ defined by

$$\gamma(t) = \frac{n + (a_n 2^n t \bmod 1)}{n(n+1)} \text{ for } \frac{2^{n-1}-1}{2^{n-1}} \leqslant t < \frac{2^n - 1}{2^n}, \quad \gamma(1) = 0.$$

We can see that this loop travels the $\left[\frac{1}{n+1}, \frac{1}{n}\right]$ segment $|a_n|$ times (left to right if $a_n > 0$ and right to left if $a_n < 0$) in the time between $\frac{2^{n-1}-1}{2^{n-1}}$ and $\frac{2^n-1}{2^n}$ for all n so indeed $g(\gamma) = a$ as desired. Thus, $\pi_1(X/A)$ is not countably generated.

Spring 2019-8. (a) Show that any continuous map $\mathbb{RP}^2 \to S^1 \times S^1$ is nullhomotopic.
(b) Find, with proof, a continuous map $S^1 \times S^1 \to \mathbb{RP}^2$ that is not nullhomotopic.

Hint: $\pi_1(\mathbb{RP}^2) = \mathbb{Z}/2$ while $\pi_1(S^1 \times S^1) = \mathbb{Z}^2$ so $f_*(\pi_1(\mathbb{RP}^2)) = 1$ so we can lift to the universal cover $\mathbb{R}^2$ which we can then homotope to 0. The map takes the first coordinate of $S^1 \times S^1$ to the equator of $\mathbb{RP}^2$ (with antipodal identification). Standard loop in S^1 gets pushed forward to a loop with non-trivial homotopy in $\mathbb{RP}^2$.

(a) Since $\pi_1(\mathbb{RP}^2, *) = \mathbb{Z}/2$ and $\pi_1(S^1 \times S^1, *) = \mathbb{Z} \times \mathbb{Z}$, we know that $f_* : \pi_1(\mathbb{RP}^2, *) \to \pi_1(S^1 \times S^1, *)$ is trivial because the trivial subgroup is the only finite subgroup of $\mathbb{Z}^2$. Hence, $f_*(\pi_1(\mathbb{RP}^2, *)) = 0 \subset p_*(\pi_1(\mathbb{R}^2, *))$ where $p : \mathbb{R}^2 \to S^1 \times S^1$ is the universal cover so the map f lifts to a map $\widetilde{f} : \mathbb{RP}^2 \to \mathbb{R}^2$. Now, let $f_t : \mathbb{RP}^2 \to S^1 \times S^1$ be defined by $f_t = p \circ (t\widetilde{f})$. Clearly, this is continuous and $f_0 = p(0)$ is constant while $f_1 = p\widetilde{f} = f$ so f is nullhomotopic.

(b) Consider $f : S^1 \times S^1 \to \mathbb{RP}^2$ defined by

$$(e^{i\theta}, e^{i\phi}) \mapsto e^{i\theta/2},$$

where we treat $S^1/\sim$, a circle quotient the antipodal relation, to be the equator of $\mathbb{RP}^2$. Note that this is well-defined since in $S^1/\sim$, we have $e^{ia} = e^{ia+i\pi} = -e^{ia}$ for any $a \in \mathbb{R}$ because we identify antipodal points.

Now, let $\gamma : [0,1] \to S^1 \times S^1$ be the loop given by $\gamma(t) = (e^{2\pi i t}, 1)$. Then, $f_*(\gamma)$ is a path that (before the antipodal identification) goes from a point to its antipodal point on the equator of $\mathbb{RP}^2$. So $f_*(\gamma)$ is not the identity of $\pi_1(\mathbb{RP}^2)$, showing that f cannot be nullhomotopic since f_* is not the trivial map.

Spring 2019-9. Let W be the space obtained by attaching two 2-cells to S^1, one by the map $z \mapsto z^4$ and the other by the map $z \mapsto z^7$. (Here $z = e^{i\theta}$ is the coordinate on $S^1 = \partial D^2$.)

(a) Compute the homology groups of W with $\mathbb{Z}$ coefficients.
(b) Is W homotopy equivalent to S^2?

Hint: Use CW structure. $H_*(W) = \mathbb{Z}_{(2)} \oplus \mathbb{Z}_{(0)}$. Yes, they are equivalent. Use Hurewicz ($\pi_2(W) \cong H_2(W)$) and Whitehead ($f : S^2 \to W$ induces isomorphism on all homology groups, since it has degree 1, so is a homotopy equivalence since W is a CW complex and both simply connected).

(a) We use the CW structure of W. First S^1 is obtained by attaching one 1-cell to one 0-cell. Then, we attach the two 2-cells as described in the question. This gives the cellular chain complex

$$0 \to \mathbb{Z}^2 \xrightarrow{f} \mathbb{Z} \xrightarrow{g} \mathbb{Z} \to 0$$

where $f(a,b) = 4a + 7b$ and $g = 0$ based on the attaching maps. Since 4 and 7 are coprime, f is surjective. Taking homology of this complex gives

$$H_n(W) = \begin{cases} \mathbb{Z} & n = 0, 2, \\ 0 & \text{otherwise.} \end{cases}$$

(b) By the Hurewicz theorem, we have an isomorphism $\pi_2(W) \cong H_2(W) = \mathbb{Z}$. Let $f : S^2 \to W$ be a generator of $\pi_2(W)$ so it has degree 1 since $f_* \in H_2(W)$ is also a generator. Thus, f induces isomorphisms on all homology groups. But then W is a CW complex and both W and S^2 are simply connected so f is in fact a homotopy equivalence by Whitehead's theorem. So yes, W is homotopy equivalent to S^2.

Spring 2019-10. Suppose M^n is a compact, connected, orientable topological n-manifold with boundary a rational homology sphere, i.e., with $H_*(\partial M; \mathbb{Q}) \cong H_*(S^{n-1}; \mathbb{Q})$.

(a) Assuming n is odd, use Poincaré duality (with $\mathbb{Q}$ coefficients) to show that M has Euler characteristic $\chi(M) = 1$.
(b) Assuming $n \equiv 2 \pmod 4$, show that the Euler characteristic $\chi(M)$ is odd.

Hint: Glue $M \cup_{\partial M} M$. This has 0 Euler characteristic so use Mayer Vietoris sequence. (b) Apply Lefschetz duality to reduce to showing $\text{Rank}(H^{2k+1}(M; \mathbb{Q}))$ is even. Show this by the bilinear, anti-symmetric, non-degenerate pairing $H^{2k+1}(M;\mathbb{Q}) \times H^{2k+1}(M;\mathbb{Q}) \to \mathbb{Q}$ by $(\omega, \eta) \mapsto \int_M \omega \wedge \eta$ corresponding to a $k \times k$ matrix with $A = -A^T$ so k must be even or else $\det(A) = -\det(A)$.

(a) By the same proof of Spring 2022-9 (we don't need $\mathbb{Z}/2$ coefficients since M here is orientable), we see that $\chi(M) = \frac{1}{2}\chi(\partial M)$. But then, the Euler characteristic only depends on the homology, so we have $\chi(M) = \frac{1}{2}\chi(S^{n-1}) = \frac{1}{2}(1 + (-1)^{n-1}) = 1$ since n is odd.

(b) Let $n = 4k + 2$. By Lefschetz duality and the universal coefficient theorem, we have

$$H^n(M;\mathbb{Q}) \cong H_0(M, \partial M; \mathbb{Q}) \cong \widetilde{H}_0(M/\partial M; \mathbb{Q}) = 0,$$

$$H^0(M;\mathbb{Q}) \cong H_n(M, \partial M; \mathbb{Q}) \cong \widetilde{H}_n(M/\partial M; \mathbb{Q}) = \mathbb{Q},$$

since $M/\partial M$ is a closed orientable manifold. Consider the long exact sequence for the pair $(M, \partial M)$ with $\mathbb{Q}$ coefficients:

$$\cdots \to H_i(\partial M;\mathbb{Q}) \to H_i(M;\mathbb{Q}) \to H_i(M, \partial M;\mathbb{Q}) \to H_{i-1}(\partial M;\mathbb{Q}) \to \cdots .$$

Since ∂M is a rational homology sphere, we have

$$H_i(\partial M;\mathbb{Q}) \cong H_i(S^{n-1};\mathbb{Q}) = \begin{cases} \mathbb{Q} & i = 0, n-1 \\ 0 & \text{otherwise} \end{cases},$$

so $H_i(M;\mathbb{Q}) \cong H_i(M, \partial M;\mathbb{Q})$ for all $2 \leqslant i \leqslant n-2$. For $i = n-1$, the long exact sequence yields

$$0 \to \mathbb{Q} \to \mathbb{Q} \to H_{n-1}(M;\mathbb{Q}) \to H_{n-1}(M, \partial M;\mathbb{Q}) \to 0,$$

which implies that $H_{n-1}(M;\mathbb{Q}) \cong H_{n-1}(M, \partial M;\mathbb{Q})$ as well since everything here is just a $\mathbb{Q}$-vector space and the alternating sum of the dimensions must be 0. Then, by the universal coefficient theorem and Lefschetz duality, for all $2 \leqslant i \leqslant n-1$

$$H^i(M;\mathbb{Q}) \cong H_i(M;\mathbb{Q}) \cong H_i(M, \partial M;\mathbb{Q}) \cong H^{n-i}(M;\mathbb{Q}).$$

Hence, using the formula for Euler characteristic gives (using $n = 4k+2$ so $n-1$ is odd and n is even)

$$\begin{aligned}\chi(M) &= \sum_{i=0}^{4k+2} (-1)^i \text{Rank}(H^i(M;\mathbb{Q})) \\ &= \text{Rank}(H^0(M;\mathbb{Q})) + \text{Rank}(H^{4k+2}(M;\mathbb{Q})) + \sum_{i=1}^{4k+1} (-1)^i \text{Rank}(H^i(M;\mathbb{Q})) \\ &= 1 + 0 + 2\sum_{i=1}^{2k} (-1)^i \text{Rank}(H^i(M;\mathbb{Q})) - \text{Rank}(H^{2k+1}(M;\mathbb{Q})) \\ &= 1 + 2m - \text{Rank}(H^{2k+1}(M;\mathbb{Q})),\end{aligned}$$

so it suffices to show that $\text{Rank}(H^{2k+1}(M;\mathbb{Q}))$ is even. This is exactly Fall 2021-5

Fall 2018

Fall 2018-1. Let M be a compact smooth n-manifold, and $f : M \to \mathbb{R}^n$ a smooth map. Let

$$S = \{p \in M \mid \text{Rank}(df_p) < n\}.$$

(a) Prove $S \neq \varnothing$.
(b) Prove $f(S) \subset \mathbb{R}^n$ has empty interior.

Hint: Submersions are open maps, M compact so f closed so $f(M)$ closed and open. Sard's theorem.

(a) If $S = \varnothing$, then f is a submersion by definition. Then, we know that submersions are open maps (because they are locally projections which are open). In particular $f(M) \subset \mathbb{R}^n$ is open. Also, M is compact so $f(M) \subset \mathbb{R}^n$ is compact so is closed implying that $f(M) = \mathbb{R}^n$ since $\mathbb{R}^n$ is connected and $f(M) \neq \varnothing$. But $\mathbb{R}^n$ is not compact so we have a contradiction and S must not be empty.

(b) $f(S)$ is exactly the set of critical values of f. By Sard's theorem, $f(S)$ has measure zero in $\mathbb{R}^n$. Then, if $f(S)$ had non-empty interior $O \subset f(S)$, then we could find $x \in O$ and an open ball $B(x,r) \subset O$ for some $r > 0$, implying that the measure of $f(S)$ is larger than the measure of $B(x,r)$ which is positive, a contradiction so $f(S)$ has empty interior.

Fall 2018-2. Let M_n be the space of $n \times n$ real matrices, viewed as the smooth manifold $\mathbb{R}^{n^2}$. Let M_n^k be the subset of matrices of rank k. Prove that M_n^k is a smooth submanifold of M_n. (Hint: First prove the subset of M_n^k where the top-left $k \times k$ minor is non-singular is a smooth submanifold of M_n.)

Hint: $\begin{pmatrix} I_k & 0 \\ -CA^{-1} & I_k \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix}$ so consider $f : N \to M_{n-k}$ given by $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto D - CA^{-1}B$. Preimage theorem.

Referenced in: Spring 2015-1, Fall 2014-6, Spring 2013-1.

Let N be the subset of M_n where the top-left $k \times k$ minor is non-singular. Then, a matrix $M \in N$ is of the form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

for some $k \times k$ matrix A with $\det(A) \neq 0$ and matrices B, C, and D of the appropriate dimensions. Since the determinant is a continuous map, having top-left $k \times k$ minor non-singular is an open condition so $N \subset M_n$ is open. Now, note that

$$\begin{pmatrix} I_k & 0 \\ -CA^{-1} & I_k \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix},$$

so M has the same rank as the right matrix above since we multiply by an invertible matrix. Thus, M has rank k if and only if $D - CA^{-1}B = 0$. So consider the map

$$f : N \to M_{n-k}, \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto D - CA^{-1}B.$$

Then, $N^k \subset N$, the subset of rank k matrices is exactly the preimage of 0 under this map, $N^k = f^{-1}(0)$. Note also that f is smooth since matrix operations (including taking inverses by Cramer's rule) are smooth with respect to the coordinates of the matrices. Thus, if we show that 0 is a regular value of f, then by the preimage theorem, we know that N^k is a smooth submanifold of M_n of codimension $(n-k)^2$.

In fact, we see that f is a submersion because for any $M \in N$ and $X \in M_{n-k} = T_{f(M)}M_{n-k}$,

$$\text{let } \widetilde{X} = \begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix}, df_M(\widetilde{X}) = \lim_{t \to 0} \frac{f(M + t\widetilde{X}) - f(M)}{t} = \lim_{t \to 0} \frac{D + tX - CA^{-1}B - (D - CA^{-1}B)}{t} = X.$$

So the subset of M_n^k where the top-left $k \times k$ minor is non-singular is a submanifold of M_n. Now, we note that any rank k matrix $\widetilde{M}$ is equivalent to something in N. Namely, we can find invertible matrices A and B such that $A\widetilde{M}B \in N$ by permuting the rows and columns of $\widetilde{M}$ since $\widetilde{M}$ has a non-singular $k \times k$ minor somewhere in it.

These operations are smooth and invertible so this gives us charts for the set of all rank k matrices. Then, the transition functions are clearly smooth since the operation of matrix multiplication is smooth and so too is f. Thus, M_n^k is a smooth submanifold of M_n of dimension $n^2 - (n-k)^2 = k(2n-k)$.

Fall 2018-3. Let θ be the restriction of

$$(x^2 dx^1 - x^1 dx^2) + (x^4 dx^3 - x^3 dx^4) + \cdots + (x^{2n} dx^{2n-1} - x^{2n-1} dx^{2n})$$

to the unit sphere $S^{2n-1} \subset \mathbb{R}^{2n}$. Prove $\ker(\theta)$ is a distribution on S^{2n-1}. Is it integrable?

Hint: Treat θ as a matrix, it always has rank 1 so kernel always has dimension $n-1$ so is a distribution. Not integrable by Frobenius since $\theta \wedge d\theta \neq 0$.

For $\ker\theta$ to be a distribution, it suffices to show θ is nonvanishing, because then the pointwise dimension of $\ker\theta$ is constant. Note that θ can be represented by the following $1 \times 2n$ matrix:

$$\theta = \begin{pmatrix} x^2 & -x^1 & \cdots x^{2n} & -x^{2n-1} \end{pmatrix},$$

which clearly has rank at most 1. In particular, θ has rank 0 if and only if $x^i = 0$ for all i which is impossible because $x \in S^{2n-1}$ so θ indeed has rank 1 on all of S^{2n-1}. Hence, θ is nonvanishing on S^{2n-1}, making $\ker(\theta)$ a distribution.

We compute

$$d\theta = (dx^2 \wedge dx^1 - dx^1 \wedge dx^2) + \cdots + (dx^{2n} \wedge dx^{2n-1} - dx^{2n-1} \wedge dx^{2n}) = 2\sum_{i=1}^{n} dx^{2i} \wedge dx^{2i-1}.$$

So that

$$\theta \wedge d\theta = 2\sum_{i=1}^{n} \sum_{j \neq i} x^{2i} dx^{2i-1} \wedge dx^{2j} \wedge dx^{2j-1} - x^{2i-1} \wedge dx^{2i} \wedge dx^{2j} \wedge dx^{2j-1}.$$

Then, by Frobenius' theorem, if $\ker(\theta)$ is integrable, then $\theta \wedge d\theta = 0$ (see for example Fall 2022-4). But $\theta \wedge d\theta = 0$ if and only if $x^i = 0$ for all i which is again impossible for $x \in S^{2n-1}$ so $\ker(\theta)$ is not integrable.

Fall 2018-4. Let M be a compact smooth 3-manifold and $\omega \in \Omega^1(M)$ a nowhere zero 1-form, so that $\ker(\omega)$ is an integrable distribution. Prove the following.

(a) $\omega \wedge d\omega = 0$.
(b) There exists some 1-form α with $d\omega = \alpha \wedge \omega$.
(c) $d\alpha \wedge \omega = 0$.

Hint: (b) $\ker(\omega) = \text{span}(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}), \omega|_U = f_U dx^m$. Set $\alpha|_U = \frac{df_U}{f_U}$ and put together with partition of unity.

In fact, we don't need compact or dimension 3. (a) and (b) are done in Fall 2022-4. (c) is simply

$$0 = d(d\omega) = d(\alpha \wedge \omega) = d\alpha \wedge \omega - \alpha \wedge d\omega = d\alpha \wedge \omega - \alpha \wedge \alpha \wedge d\omega = d\alpha \wedge \omega.$$

Fall 2018- 5. Let $M \subset \mathbb{R}^n$ be a compact $(n-1)$-dimensional submanifold, let $\iota : M \hookrightarrow \mathbb{R}^n$ be the inclusion map, and let $D \subset \mathbb{R}^n$ be the n-dimensional compact region with $\partial D = M$. Let $dV = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n \in \Omega^n(\mathbb{R}^n)$ be the standard volume form on $\mathbb{R}^n$.

(a) Define $dA \in \Omega^{n-1}(M)$, the standard volume form on M, induced by the embedding ι.
(b) Prove $\iota^*(i_X dV) = \langle X, N\rangle dA$, for any smooth vector field X on $\mathbb{R}^n$. (Here, N is the unit normal vector along M, pointing outward from D.)
(c) Prove
$$\int_D L_X(dV) = \int_M \langle X, N\rangle dA.$$
(d) Derive Gauss' Divergence theorem from the case $n = 3$.

Hint: $dA = \iota^*(i_N(dV))$. (b) $T = X - \langle X, N\rangle N$ is tangent to M so any $T, d\iota X_1, \ldots, d\iota X_{n-1}$ is linearly dependent so $\iota^*(i_T(dV))(X_1, \ldots, X_{n-1}) = 0$. (c) Stokes' and Cartan. (d) Let $X = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} + h\frac{\partial}{\partial z}$ and compute.

Referenced in: Fall 2008-5.

(a) $dA = \iota^*(i_N(dV))$ where $i_N : \Omega^n(\mathbb{R}^n) \to \Omega^{n-1}(\mathbb{R}^n)$ is the interior product, $(i_N\omega)(X_1, \ldots, X_{n-1}) = \omega(N, X_1, \ldots, X_{n-1})$.

(b) Let $T = X - \langle X, N\rangle N$. Then T is tangent to M because we have subtracted off the projection onto the normal space. So
$$\iota^*(i_T(dV))(X_1, \ldots, X_{n-1}) = dV(T, d\iota X_1, \ldots, d\iota X_{n-1}) = 0$$
since these vector fields are all tangent to M and therefore are linearly dependent since there are n of them while M is only $(n-1)$-dimensional. Thus,

$$\begin{aligned} 0 = \iota^*(i_T(dV)) = \iota^*(i_X(dV) - i_{\langle X,N\rangle N}(dV)) &= \iota^*(i_X(dV)) - \langle X, N\rangle \iota^*(i_N(dV)) \\ &= \iota^*(i_X(dV)) - \langle X, N\rangle dA, \end{aligned}$$

implying that $\iota^*(i_X(dV)) = \langle X, N\rangle dA$.

(c) Using part (b), Stokes' theorem, Cartan's formula and the fact that $d(dV) = 0$, we have

$$\int_M \langle X, N\rangle dA = \int_M \iota^*(i_X(dV)) = \int_D d(i_X(dV)) = \int_D (L_X - i_X d)(dV) = \int_D L_X(dV).$$

(d) This was done in Fall 2019-1 but we present another solution here that builds on the previous parts. Let

$X = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} + h\frac{\partial}{\partial z}$ so we have

$$\begin{aligned} L_X(dV) &= L_X(dx) \wedge dy \wedge dz + dx \wedge L_X(dy) \wedge dz + dz \wedge dy \wedge L_X(dz) \\ &= d(L_X(x)) \wedge dy \wedge dz + dx \wedge d(L_X(y)) \wedge dz + dx \wedge dy \wedge d(L_X(z)) \\ &= df \wedge dy \wedge dz + dx + dx \wedge dg \wedge dz + dx \wedge dy \wedge dh \\ &= \frac{\partial f}{\partial x} dx \wedge dy \wedge dz + \frac{\partial g}{\partial y} dx \wedge dy \wedge dz + \frac{\partial h}{\partial z} dx \wedge dy \wedge dz \\ &= \operatorname{div}(X) dV. \end{aligned}$$

Hence, by part (c),

$$\int_D \operatorname{div}(X) dV = \int_D L_X(dV) = \int_M \langle X, N \rangle dA.$$

Fall 2018-6. Can a finite rank free group have a finite index subgroup of smaller rank?

Hint: Fundamental group of wedge sum of n circles. k-fold covering. Contract to wedge sum of $k(n-1)+1$ circles.

No this is not possible. Let G be a free group of rank n and suppose H is a finite index subgroup of G with index k. Then, $G = \pi_1(X, *)$ where $X = \bigvee_{1 \leqslant i \leqslant n} S^1$ is the wedge sum of n circles. Then H is the fundamental group of a connected k-fold covering of X, say $p : \widetilde{X} \to X$ where $\widetilde{X}$ is a graph with k vertices and nk edges.

Since $\widetilde{X}$ is connected, we may take a connected subgraph of $\widetilde{X}$ consisting of all k vertices and $k-1$ edges. If we quotient $\widetilde{X}$ by this subgraph, we are left with a graph with one vertex and $nk - k + 1 = k(n-1) + 1$ edges, i.e., $\bigvee_{1 \leqslant i \leqslant k(n-1)+1} S^1$. This quotienting is a homotopy equivalence so H is isomorphic to the fundamental group of the wedge sum of $k(n-1)+1$ circles. Namely, H is a free group of rank $k(n-1)+1$ where $k \geqslant 1$. So $k(n-1)+1 \geqslant n$ and H does not have smaller rank than G.

Fall 2018-7. Prove that the covering map $S^n \to \mathbb{RP}^n$ induces an isomorphism on de Rham cohomology if and only if n is odd. What is the orientable double cover of $\mathbb{RP}^n$?

Hint: Use known cohomology groups for $\mathbb{RP}^n$ and S^n. Finite covering maps always induce injections. Double cover is S^n or $\mathbb{RP}^n \sqcup \mathbb{RP}^n$ for n even/odd respectively.

We recall

$$H^k_{dR}(S^n) = \begin{cases} \mathbb{R} & k = 0, n \\ 0 & \text{otherwise.} \end{cases} \quad \text{and } H^k_{dR}(\mathbb{RP}^n) = \begin{cases} \mathbb{R} & k = 0 \text{ or } k = n \text{ and } n \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

So if n is even, there isn't an isomorphism $H^n(\mathbb{RP}^n) \to H^n(S^n)$ so the covering map cannot induce an isomorphism on de Rham cohomology. If n is odd, then we note that $p^* : H^k_{dR}(\mathbb{RP}^n) \to H^k_{dR}(S^n)$ is an injection since p is a finite covering map so has a retraction (see for example Spring 2019-6). But injective maps between vector spaces of the same rank are isomorphisms so p indeed induces an isomorphism on de Rham cohomology if n is odd.

The orientation double cover is a 2-fold cover such that the non-trivial deck transformation reverses the orientation. For n even, S^n is the orientation double cover since the antipodal deck transformation $x \mapsto -x$ has degree $(-1)^{n+1} = -1$ so reverses orientation. For n odd, $\mathbb{RP}^n$ is oriented so the double cover is $\mathbb{RP}^n \sqcup \mathbb{RP}^n$ where the two copies have opposite orientations.

Fall 2018-8. Assume the integral homology of a space is $\mathbb{Z}$ in grading 0, $\mathbb{Z}$ in grading 2, $\mathbb{Z}/2$ in grading 3, and 0 in all other gradings.

(a) What is its integral cohomology group?
(b) Construct a simply connected CW complex X with the given homology.
(c) Construct another simply connected CW complex Y with the same homology, which is not homotopy equivalent to X.

Hint: Universal coefficient theorem. Ext of projective (free) module is 0. One $0, 2, 3, 4$-cell each. Wedge sum with acyclic space (such as $S^1 \vee S^1$ with two 2-cells attached via $a^5 b^{-3}$ and $b^3(ab)^{-2}$).

(a) The universal coefficient theorem states that for R a principal ideal domain, G and R-module, and $n \geqslant 0$, we have

$$H^n(X; G) \cong \operatorname{Hom}_R(H_n(X; R), G) \oplus \operatorname{Ext}^1_R(H_{n-1}(X; R), G).$$

So in our case, we compute:

$$H^0(X; \mathbb{Z}) \cong \operatorname{Hom}_{\mathbb{Z}}(H_0(X; \mathbb{Z}), \mathbb{Z}) \oplus \operatorname{Ext}^1_{\mathbb{Z}}(H_{-1}(X; \mathbb{Z}), \mathbb{Z}) = \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \oplus \operatorname{Ext}^1_{\mathbb{Z}}(0, \mathbb{Z}) = \mathbb{Z} \oplus 0 = \mathbb{Z},$$
$$H^1(X; \mathbb{Z}) \cong \operatorname{Hom}_{\mathbb{Z}}(H_1(X; \mathbb{Z}), \mathbb{Z}) \oplus \operatorname{Ext}^1_{\mathbb{Z}}(H_0(X; \mathbb{Z}), \mathbb{Z}) = \operatorname{Hom}_{\mathbb{Z}}(0, \mathbb{Z}) \oplus \operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) = 0 \oplus 0 = 0,$$
$$H^2(X; \mathbb{Z}) \cong \operatorname{Hom}_{\mathbb{Z}}(H_2(X; \mathbb{Z}), \mathbb{Z}) \oplus \operatorname{Ext}^1_{\mathbb{Z}}(H_1(X; \mathbb{Z}), \mathbb{Z}) = \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \oplus \operatorname{Ext}^1_{\mathbb{Z}}(0, \mathbb{Z}) = \mathbb{Z} \oplus 0 = \mathbb{Z},$$
$$H^3(X; \mathbb{Z}) \cong \operatorname{Hom}_{\mathbb{Z}}(H_3(X; \mathbb{Z}), \mathbb{Z}) \oplus \operatorname{Ext}^1_{\mathbb{Z}}(H_2(X; \mathbb{Z}), \mathbb{Z}) = \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}) \oplus \operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) = 0 \oplus 0 = 0,$$
$$H^4(X; \mathbb{Z}) \cong \operatorname{Hom}_{\mathbb{Z}}(H_4(X; \mathbb{Z}), \mathbb{Z}) \oplus \operatorname{Ext}^1_{\mathbb{Z}}(H_3(X; \mathbb{Z}), \mathbb{Z}) = \operatorname{Hom}_{\mathbb{Z}}(0, \mathbb{Z}) \oplus \operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}) = 0 \oplus \mathbb{Z}/2 = \mathbb{Z}/2.$$

Also, it is clear that $H^k(X; \mathbb{Z}) = 0$ for all $k > 4$ so to summarize, we have

$$H^k(X; \mathbb{Z}) = \begin{cases} \mathbb{Z} & k = 0, 2, \\ \mathbb{Z}/2 & k = 4, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Construct X as follows. Start with one 0-cell. Attach one 2-cell in the obvious way. Attach one 3-cell to the 0-cell in the obvious way. Finally, attach one 4-cell to the 3-cell by a degree 2 map.

(c) Let Y have one 0-cell, one 2-cell and one 3-cell attached in the same way as X. Then, give Y one 4-cell attached by a map which first does the Hopf fibration $S^3 \to S^2$ and then does a degree two map $S^3 \to S^3$. Since homology only cares about adjacent dimensions, Y has the same homology groups as X. However, they are not homotopy equivalent because $\pi_3(S^2 \vee S^3) \cong \mathbb{Z}^2$ generated by the identity and the Hopf fibration so these two attaching maps are not homotopy equivalent.

Fall 2018-9. Let X be a connected CW-complex. Show that there is a natural isomorphism

$$\widetilde{H}_k(\Sigma X; M) \cong \widetilde{H}_{k-1}(X; M)$$

for all k and for all abelian groups M.

Hint: From Mayer Vietoris, get $S^m(f)_* \sim S^{m-1}(f)_*$ and $S^m(f)_* \sim \Sigma^m(f)_*$ in commutative diagrams.

This is shown in Fall 2022-10.

Fall 2018-10. Let Y be a connected and simply connected CW-complex.

(a) Compute the fundamental group of $Y \vee S^1$.
(b) Describe the universal cover of $Y \vee S^1$, together with the action of the deck transformations.
(c) Describe all finite covers of $Y \vee S^1$, again with the action of the deck transformations.
(d) Describe what changes in the first two parts for $Y = \mathbb{RP}^2$.

Hint: Van Kampen's. Glue a copy of Y to each integer in $\mathbb{R}$. Same but with circle and k copies. $\mathbb{Z}/2 * \mathbb{Z}$. Infinite fractal of $\mathbb{R}$'s with S^2's at every integer point.

(a) By Van Kampen's theorem, $\pi_1(Y \vee S^1) = \pi_1(Y) * \pi_1(S^1) = 1 * \mathbb{Z} = \mathbb{Z}$, since Y is simply connected so has trivial fundamental group.

(b) Since Y is simply connected, it is its own universal cover while the universal cover of S^1 is $\mathbb{R}$. Thus, the universal cover of $Y \vee S^1$ is $\mathbb{R}$ with a copy of Y attached via the base point of Y to each integer in $\mathbb{R}$. Then, the deck transformations are simply given by the deck transformations of $\mathbb{R} \to S^1$ which are the translations by an integer (that cycle the copies of Y around).

(c) Similarly to part (b), a finite k-fold cover of $Y \vee S^1$ is $\mathbb{R}/k\mathbb{R} = S^1$ with a copy of Y attached via the base point of Y to each integer in $\mathbb{Z}/k\mathbb{Z}$. I.e., we get a circle with k integer points and a copy of Y glued to each one of these by its base point. The deck transformations similarly are translation by integers (mod k) which cycle the copies of Y around the circle.

(d) Now, Van Kampen's tells us that $\pi_1(Y \vee S^1) = \pi_1(\mathbb{RP}^2) * \pi_1(S^1) = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}$ which is the group $\langle a, b \mid a^2 \rangle$. We know the universal cover of $\mathbb{RP}^2$ is the double cover $S^2 \to \mathbb{RP}^2$. So the universal cover can be described as follows. Start with a copy of $\mathbb{R}$. At each integer point, attach a copy of S^2 by its north pole. At the south pole of each of these S^2's, attach a copy of $\mathbb{R}$. Repeat the process for each of these copies of $\mathbb{R}$ and so on infinitely.

The deck transformations then act as follows. Each element of $\mathbb{Z}$ translates the points in the corresponding copy of $\mathbb{R}$, and the nontrivial element of $\mathbb{Z}/2$ takes points to their antipodal point on S^2. Then, the words that are formed are these functions composed together in order of concatenation.

Spring 2018

Spring 2018-1. Suppose that M and N are connected smooth manifolds of the same dimension and $f : M \to N$ is a smooth submersion.

(a) Prove that if M is compact, then f is onto and f is a covering map.
(b) Give an example of a smooth submersion $f : M \to N$ such that M and N have the same dimension, N is compact, and f is onto, but f is not a covering map.

Hint: Submersion is open map so onto. Local diffeomorphism implies locally even covering and then piece together using compact/connected. $e^{2\pi it} : (-0.1, 1.1) \to S^1$.

Referenced in: Fall 2020-9, Fall 2014-1, Fall 2011-3, Spring 2010-3, Spring 2009-2.

(a) Since f is a submersion, it is an open map (as it is locally a projection and projections are open), so in particular $f(M)$ is open. If M is compact, then so too is $f(M)$ implying that $f(M)$ is closed since N is Hausdorff. Then, N is connected and $f(M) \neq \varnothing$ is both open and closed so $f(M) = N$ and f is surjective.

As a submersion between manifolds of the same dimension, f is locally a diffeomorphism by the inverse function theorem since df_x is an isomorphism for any $x \in M$. For any $p \in N$, consider $f^{-1}(p) \subset M$. Since $\{p\} \subset N$ is closed, so is $f^{-1}(p)$ implying that $f^{-1}(p)$ is compact since M is compact. By the preimage theorem, since f is a submersion, $f^{-1}(p)$ is a zero dimensional submanifold of M which means it is a collection of isolated points. But since it is compact, it must be a finite set, namely $f^{-1}(p) = \{p_1, \ldots, p_n\}$.

Then, for each $p_i \in f^{-1}(p)$, choose a neighborhood $U_i \ni p_i$ so that $f|_{U_i}$ is a diffeomorphism. Shrink the U_i's as necessary so that $\overline{U}_i \cap \overline{U}_j = \varnothing$ for all $i \neq j$. Define $p \in W = \bigcap_{i=1}^n f(U_i) - f(M - \bigcup_{i=1}^n U_i) \subset N$ and $W_i = f^{-1}(W) \cap U_i$. By construction, we have $f^{-1}(W) \subset \bigcup_{i=1}^n U_i$ so $f^{-1}(W) = \bigsqcup_{i=1}^n W_i$ and $f|_{W_i} : W_i \to W$ is a diffeomorphism so f locally is an even covering of W.

So cover N by W_p for $p \in N$ so that f evenly covers each W_p. It suffices now to show that the size of $f^{-1}(p)$ is constant. Since M is compact, $F(M) = N$ is compact so we reduce this to a finite cover. On any $q \in W_p$, we note that the W_i's which evenly cover W_p also act as an even cover for the neighborhood W_p of q so the size of $f^{-1}(p)$ locally does not depend on p. Thus, since N is connected, we can get between any two points by a finite chain of these opens, on each of which, the size of $f^{-1}(p)$ is constant so it is constant on all of N by induction. Thus f is a covering map.

(b) Let $M = (-0.1, 1.1) \subset \mathbb{R}, N = S^1$, and $f(t) = e^{2\pi i t}$. Both M and N are connected 1-manifolds and N is compact. Also f is a submersion since $df_t = 2\pi i e^{2\pi i t}$ is never zero. Moreover, f is onto since $S^1 = \{e^{i\theta} \in \mathbb{C} \mid 0 \leqslant \theta < 2\pi\}$. But f is not a covering map because $f^{-1}(1) = \{0, 1\}$ while $f^{-1}(-1) = \{0.5\}$ which do not have the same size.

Spring 2018-2. Let $\Phi_N, \Phi_S : \mathbb{R} \times S^2 \to S^2$ be two global flows on the sphere S^2. Show that there exist $\epsilon > 0$, a neighborhood U of the North pole, a neighborhood V of the South pole, and a global flow $\Phi : \mathbb{R} \times S^2 \to S^2$ such that $\Phi(t, q) = \Phi_N(t, q)$ for all $t \in (-\epsilon, \epsilon), q \in U$, and $\Phi(t, q) = \Phi_S(t, q)$ for all $t \in (-\epsilon, \epsilon), q \in V$.

Hint: Partition of unity subordinate to $S^2 - \overline{U}, S^2 - \overline{V}$. Use corresponding vector fields and fundamental theorem of flows guarantees uniqueness.

Let U and V be any neighborhoods of the north and south pole respectively that are small enough so that the open sets $S^2 - \overline{U}$ and $S^2 - \overline{V}$ cover S^2. Then, let λ, ψ be a partition of unity subordinate to this cover. λ is supported on $S^2 - \overline{U}$ so $\lambda|_U \equiv 0$ implying that $\psi|_U \equiv 1$ and similarly $\lambda|_V \equiv 1$.

Let X and Y be the vector fields corresponding to Φ_N and Φ_S respectively. Define $Z = \psi X + \lambda Y$ and note that $Z|_U \equiv X$ and $Z|_V \equiv Y$. Since S^2 is compact, any smooth vector field generates a global flow so we let Φ be the global flow generated by Z. By the fundamental theorem of ODEs, on U, this flow is unique for a short time interval $t \in (-\epsilon_1, \epsilon_1)$ and similarly on V, this flow is unique for $t \in (-\epsilon_2, \epsilon_2)$.

So we know that Φ agrees with Φ_N on U for $t \in (-\epsilon_1, \epsilon_1)$ by uniqueness since Z and X agree on U, and similarly for Φ_S. Let $\epsilon = \min(\epsilon_1, \epsilon_2)$. Then, we have the desired property.

Spring 2018-3. For $n \geqslant 1$, consider the subset $X \subset \mathbb{CP}^{2n}$ given by

$$X = \{[z_0 : z_1 : \cdots : z_{2n}] \in \mathbb{CP}^{2n} \mid z_{n+1} = z_{n+2} = \cdots = z_{2n} = 0\}.$$

(a) Show that X is a smooth submanifold.
(b) Calculate the mod 2 intersection number of X with itself.

Hint: Obvious inclusion is an injective proper immersion. $f : [z_0 : \cdots : z_n] \mapsto [0 : \cdots : 0 : z_n : \cdots : z_0]$ is homotopic to inclusion and is transverse to X. Only one point in intersection.

(a) Define

$$i : \mathbb{CP}^n \to \mathbb{CP}^{2n}, \quad [z_0 : \cdots : z_n] \mapsto [z_0 : \cdots : z_n : 0 : \cdots : 0].$$

It is evident that i is a bijection onto its image which is exactly X and that i is smooth. We know that $\mathbb{CP}^n$ is compact so i is automatically proper. This is because if $K \subset \mathbb{CP}^{2n}$ is compact, then K is closed (as manifolds are Hausdorff) so $i^{-1}(K)$ is closed so is compact as a closed subset of a compact space.

Hence it suffices to show that i is an immersion as the image of an injective proper immersion is an embedded smooth submanifold. Let $p \in \mathbb{CP}^n$ with $p = [z_0 : \cdots : z_n]$. Some $z_j \neq 0$ so consider the charts

$$p \in U_j^n = \{[z_0 : \cdots : z_n] \in \mathbb{CP}^n \mid z_j \neq 0\}, \quad [z_0 : \cdots : z_n] \mapsto \left(\frac{z_0}{z_j}, \cdots, \widehat{z_j}, \cdots, \frac{z_n}{z_j}\right)$$

$$i(p) \in U_j^{2n} = \{[z_0 : \cdots : z_{2n}] \in \mathbb{CP}^{2n} \mid z_j \neq 0\}, \quad [z_0 : \cdots : z_{2n}] \mapsto \left(\frac{z_0}{z_j}, \cdots, \widehat{z_j}, \cdots, \frac{z_{2n}}{z_j}\right).$$

With respect to these charts, it is clear that i takes the form

$$\left(\frac{z_0}{z_j}, \cdots, \widehat{z_j}, \cdots, \frac{z_n}{z_j}\right) \mapsto \left(\frac{z_0}{z_j}, \cdots, \widehat{z_j}, \cdots, \frac{z_n}{z_j}, 0, \cdots, 0\right),$$

so i is indeed an immersion and thus a smooth embedding of $\mathbb{CP}^n$ into $\mathbb{CP}^{2n}$.

(b) We seek to compute $I_2(i, X)$. To do this, we first define

$$f : \mathbb{CP}^n \to \mathbb{CP}^{2n}, \quad [z_0 : \cdots : z_n] \mapsto [0 : \cdots : 0 : z_n : \cdots : z_0],$$

and consider the homotopy

$$H : \mathbb{CP}^n \times [0,1] \to \mathbb{CP}^{2n}, \quad ([z_0 : \cdots : z_n], t) \mapsto [tz_0 : \cdots : tz_{n-1} : z_n : (1-t)z_{n-1} : \cdots : (1-t)z_0].$$

So $H(z, 0) = f(z)$ while $H(z, 1) = i(z)$ and H is evidently smooth so i and f are homotopic. Moreover, $X \cap f(\mathbb{CP}^n)$ is the singleton $\{p\}$ where $p = [0 : \cdots 0 : 1 : 0 : \cdots : 0]$ with the 1 in position $n + 1$. Then, $T_p(X) \subset T_p\mathbb{CP}^{2n}$ consists of vectors whose last $2n$ real coordinates are zero, while $df(T_{[0:\cdots:0:1]}\mathbb{CP}^n)$ consists of vectors whose first $2n$ real coordinates are zero. So, we have

$$T_p\mathbb{CP}^{2n} = T_pX + df(T_{[0:\cdots:0:1]}\mathbb{CP}^n),$$

showing that f is transverse to X. Hence, we may compute $I_2(f, X)$ which will be equal to $I_2(i, X) = I_2(X, X)$ since i and f are homotopic. But then $X \cap f(\mathbb{CP}^n)$ is a single point so $I_2(f, X) = 1$.

Spring 2018-4. Suppose N is a smoothly embedded submanifold of a smooth manifold M. A vector field X on M is called tangent to N if $X_p \in T_pN \subset T_pM$ for all $p \in N$.

(a) Show that if X and Y are vector fields on M both tangent to N, then $[X, Y]$ is also tangent to N.
(b) Illustrate this principle by choosing two vector fields X, Y tangent to $S^2 \subset \mathbb{R}^3$ (such that $[X, Y]$ is not identically zero), computing $[X, Y]$ and checking that it is tangent to S^2.

Hint: Compute using local coordinates for a slice chart and sum notation. $X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, Y = -z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}, [X, Y] = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}$.

(a) We use local coordinates. Let $x^1, \ldots, x^n$ be local coordinates for N and extend to $x^1, \ldots, x^n, \ldots, x^m$ local coordinates for M using a slice chart. So we can express X and Y as

$$X = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}, \quad Y = \sum_{i=1}^n Y^i \frac{\partial}{\partial x^i},$$

for some $X^i, Y^i \in C^\infty(N)$. Then, we may simply compute

$$\begin{aligned}
[X, Y] &= XY - YX \\
&= \sum_{i=1}^n X^i \frac{\partial}{\partial x^i} \sum_{j=1}^n Y^j \frac{\partial}{\partial x^j} - \sum_{i=1}^n Y^i \frac{\partial}{\partial x^i} \sum_{j=1}^n X^j \frac{\partial}{\partial x^j} \\
&= \sum_{i=1}^n \sum_{j=1}^n X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial}{\partial x^j} - \sum_{i=1}^n \sum_{j=1}^n Y^i \frac{\partial X^j}{\partial x^i} \frac{\partial}{\partial x^j} \\
&= \sum_{j=1}^n \left(\sum_{i=1}^n X^i \frac{\partial Y^j}{\partial x^i} - \sum_{i=1}^n Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j},
\end{aligned}$$

which is still a vector field in N since $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$ span the tangent space to N and $X^i, \frac{\partial Y^j}{\partial x^i}, Y^i, \frac{\partial X^j}{\partial x^i} \in C^\infty(N)$.

(b) Let $X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ and $Y = -z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}$. Then,

$$[X, Y] = (X(-z) - Y(-y))\frac{\partial}{\partial x} + (X(0) - Y(x))\frac{\partial}{\partial y} + (X(x) - Y(0))\frac{\partial}{\partial z} = 0\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z},$$

which is still tangent to S^2 as desired.

Spring 2018-5. A symplectic form on an eight-dimensional manifold is defined to be a closed two-form ω such that $\omega \wedge \omega \wedge \omega \wedge \omega$ is a volume form (that is, everywhere nonvanishing). Determine which of the following manifolds admit symplectic forms: (a) S^8; (b) $S^2 \times S^6$; (c) $S^2 \times S^2 \times S^2 \times S^2$.

Hint: Only $S^2 \times S^2 \times S^2 \times S^2$, form is $\alpha + \beta + \gamma + \delta \in H^2$. Use the cohomology ring of S^n and Künneth's formula.

First, recall the cohomology groups and cohomology ring of S^n are:

$$H^k_{dR}(S^n) = \begin{cases} \mathbb{R} & k = 0, n, \\ 0 & \text{otherwise.} \end{cases}, \quad H^*(S^n; \mathbb{Z}) = \mathbb{Z}[\alpha]/(\alpha^2), \quad |\alpha| = n.$$

We claim S^8 and $S^2 \times S^6$ cannot have a symplectic form. Suppose ω was a symplectic form on S^8. Then, since $H^2_{dR}(S^8) = 0$, we know that ω is exact so there exists $\eta \in \Omega^1(S^8)$ such that $d\eta = \omega$. In which case $\omega^4 = \omega \wedge \omega \wedge \omega \wedge \omega$ is also exact with $d(\eta \wedge \omega \wedge \omega \wedge \omega) = \omega^4$ since $d\omega = 0$. But then

$$\int_{S^8} \omega^4 = \int_{S^8} d(\eta \wedge \omega \wedge \omega \wedge \omega) = \int_{\partial S^8} \eta \wedge \omega \wedge \omega \wedge \omega = 0$$

by Stokes' theorem and since S^8 has no boundary. Thus ω^4 cannot be a volume form so ω is not a symplectic form.

For $S^2 \times S^6$, Künneth's formula (and de Rham's theorem) tells us that

$$H^*_{dR}(S^2 \times S^6) \cong H^*_{dR}(S^2) \otimes H^*_{dR}(S^6) \cong \mathbb{R}[\alpha, \beta]/(\alpha^2, \beta^2), \quad |\alpha| = 2, |\beta| = 6.$$

Thus, the elements in grading 2 are all of the form $x\alpha$ for some $x \in \mathbb{R}$. So our closed 2-form ω is of the form $x\alpha$ meaning that $\omega \wedge \omega = (x\alpha)^2 = x^2\alpha^2 = 0$ on the level of cohomology. This again means that ω^4 is exact and so cannot be a volume form.

Finally, we claim that $S^2 \times S^2 \times S^2 \times S^2$ does have a symplectic form. Here, we have

$$H^*_{dR}(S^2 \times S^2 \times S^2 \times S^2) \cong \mathbb{R}[\alpha, \beta, \gamma, \delta]/(\alpha^2, \beta^2, \gamma^2, \delta^2), \quad |\alpha| = |\beta| = |\gamma| = |\delta| = 2.$$

Then, take the element $\omega = \alpha + \beta + \gamma + \delta \in H^*_{dR}(S^2 \times S^2 \times S^2 \times S^2)$ which is in grading 2. Then, we can see that $\omega^4 = y\alpha\beta\gamma\delta$ for some nonzero coefficient $y \in \mathbb{R}$ by the quotient structure of the cohomology ring. This ω^4 is thus a generator of $H^8_{dR}(S^2 \times S^2 \times S^2 \times S^2) \cong \mathbb{R}$ so is a volume form and ω is the desired symplectic form.

Spring 2018-6. Let U be a bounded open set in $\mathbb{R}^3$ with smooth boundary, and let V be a smooth vector field on $\mathbb{R}^3$. The classical divergence theorem expresses the triple integral $\iiint_V \operatorname{div}(V) d(\text{vol})$ as a surface integral over the boundary of V. State this theorem, and show how it can be obtained as a particular case of Stokes' Theorem for differential forms.

Hint: $\iiint_M \operatorname{div}(X) dV = \iint_{\partial M} \langle X, n \rangle dS$ for n the outward pointing normal unit vector to ∂M and $dS = i^* \iota_n dV$ the induced volume form on ∂M. Recall $\operatorname{div}(X) dV = L_X(dV)$ and use Cartan's magic formula. Consider $Y = X - \langle X, n \rangle n$.

This is exactly Fall 2019-1.

Spring 2018-7. Let M and N be smooth, connected, orientable n-manifolds for $n \geqslant 3$, and let $M \# N$ denote their connected sum.

(a) Compute the fundamental group of $M \# N$ in terms of that of M and of N (you may assume that the basepoint is on the boundary sphere along which we glue M and N).
(b) Compute the homology groups of $M \# N$. (You may use without proof that $H_n(-; \mathbb{Z})$ of a connected orientable n-manifold is always isomorphic to $\mathbb{Z}$).
(c) For part (a), what changes if $n = 2$? Use this to describe the fundamental groups of orientable surfaces.

Hint: Van Kampen's gives $\pi_1(M\#N) \cong \pi_1(M) * \pi_1(N)$. Good pair with gluing sphere S^{n-1} gives direct sum for $i \neq 0, n$ in which cases it's just $\mathbb{Z}$.

Referenced in: Spring 2010-7.

(a) Suppose that M and N are glued together by the disk D^n. Let A be a neighborhood of $M - D^n$ in $M\#N$ that deformation retracts onto $M - D^n$ and let B be a neighborhood in $M\#N$ of $N - D^n$ that deformation retracts onto $N - D^n$. We can see that $A \cap B$ deformation retracts onto $\partial D^n = S^{n-1}$ and $A \cup B = M\#N$. So by Van Kampen's theorem, we have

$$\pi_1(M\#N) \cong \pi_1(A) * \pi_1(B)/N$$

where N is the normal subgroup generated by elements of the form $i_*^A(\gamma)i_*^B(\gamma)$ where $i^A : A \cap B \to A$ and $i^B : A \cap B \to B$ are the inclusion maps. Since $n \geqslant 3$, $A \cap B$ which is homotopy equivalent to S^{n-1} has trivial fundamental group so N is the trivial subgroup.

In addition, since A is homotopy equivalent to $M - D^n$, we have $\pi_1(A) \cong \pi_1(M - D^n)$. But then we can make $M - D^n$ into M by attaching an n-cell which implies that $\pi_1(M - D^n) \cong \pi_1(M)$ as $n \geqslant 3$. Similarly, $\pi_1(B) \cong \pi_1(N)$. Hence, we have $\pi_1(M\#N) \cong \pi_1(M) * \pi_1(N)$.

(b) We showed in Spring 2021-7 that

$$H_i(M\#N) = \begin{cases} \mathbb{Z} & i = 0, n, \\ H_i(M) \oplus H_i(N) & \text{otherwise.} \end{cases}$$

(c) If $n = 2$, we no longer have $\pi_1(A) \cong \pi_1(M)$ nor $\pi_1(B) \cong \pi_1(N)$. We also don't have $\pi_1(A \cap B) = 0$ so N as described above is no longer trivial. However, we know that a compact, oriented 2-manifold can be described uniquely by its genus and that a genus g torus is the connected sum of g standard (genus 1) tori. Let T_g be the genus g torus and we know

$$\pi_1(T_g) = \langle a_1, \ldots, a_{2g} \mid a_1 a_2 a_1^{-1} a_2^{-1} \cdots a_{2g-1} a_{2g} a_{2g-1}^{-1} a_{2g}^{-1} \rangle.$$

Moreover, we have $\pi_1(T_g \# T_h) \cong \pi_1(T_{g+h})$ using the above logic in part (a).

Spring 2018-8. Determine all the possible degrees of maps $S^2 \to S^1 \times S^1$.

Hint: $\pi_1(S^2) = 1$ so can lift $f : S^2 \to S^1 \times S^1$ so f is nullhomotopic so only degree 0 is possible.

Referenced in: Spring 2010-10.

Let $f : S^2 \to S^1 \times S^1$ be any map. Then, since S^2 is simply connected, we have $\pi_1(S^2, *) = 0$, so $f_*(\pi_1(S^2, *)) = 0 \subset p_*(\pi_1(\mathbb{R}^2, *))$ where $p : \mathbb{R}^2 \to S^1 \times S^1$ is the universal cover. Thus, f lifts to a map $\widetilde{f} : S^2 \to \mathbb{R}^2$. Now, let $f_t : S^2 \to \mathbb{R}^2$ be defined by $f_t = p \circ (t\widetilde{f})$. Clearly, this is continuous (in t) and $f_0 = p(0)$ is constant while $f_1 = p \circ \widetilde{f} = f$ so f is nullhomotopic. Hence, $f_* : H^2(S^2) \to H^2(S^1 \times S^1)$ is the zero map so f must have degree 0.

Spring 2018-9. Point S^2 via the south pole, and consider the Cartesian product $S^2 \times S^2$.

(a) Describe a cell structure on $S^2 \times S^2$ that is compatible with the inclusion of

$$S^2 \vee S^2 \hookrightarrow S^2 \times S^2$$

as those pairs where one coordinate is the south pole.

(b) Let X be $(S^2 \times S^2) \cup_{S^2} D^3$, where we attach the 3-disk via the map

$$S^2 \to S^2 \vee S^2$$

which crushes a great circle connecting the north and south poles. Compute the homology groups of X.

Hint: One 0-cell and one 2-cell for each S^2. Product cell structure. $H_k(X) = \mathbb{Z}_{(4)} \oplus \mathbb{Z}_{(2)} \oplus \mathbb{Z}_{(0)}$.

(a) First, note that S^2 can be given a cell structure with one 0-cell, e_0, and one 2-cell, e_2, attached in the obvious way. Let $\{e_0, e_2\}$ and $\{f_0, f_2\}$ be the cell structures of two copies of S^2 with the 0-cells being at the south poles. Hence, we can give $S^2 \times S^2$ a cell structure with one 0-cell, $e_0 \times f_0$, two 2-cells, $e_0 \times f_2$ and $e_2 \times f_0$, both attached to the 0-cell in the obvious way (with the 0-cell at the south pole), and one 4-cell $e_2 \times f_2$ that is attached to each of the two 2-cells by degree 1 maps. In particular, by this construction, the 2-skeleton is exactly $S^2 \vee S^2$ so this structure is compatible with the inclusion.

(b) By attaching the 3-cell, we have the following cell structure:

$$0 \to \mathbb{Z} \xrightarrow{f} \mathbb{Z} \xrightarrow{g} \mathbb{Z}^2 \to 0 \to \mathbb{Z}.$$

We can see that f is 0 because the 4-cell is attached to the 2-skeleton directly, not the 3-cell. Also, we can see that g sends the generator of $H_3(X)$ to $(1,1) \in \mathbb{Z}^2$ since it is attached to both copies of S^2 via degree 1 maps. Hence, we easily compute the homology to be

$$H_k(X) = \begin{cases} \mathbb{Z} & k = 0, 2, 4, \\ 0 & \text{otherwise.} \end{cases}$$

Spring 2018-10. Let X be a semi-locally simply connected space and let $\widetilde{X} \to X$ be the universal cover.

(a) Show that any map $\sigma : \Delta^n \to X$ lifts to a map $\widetilde{\sigma} : \Delta^n \to \widetilde{X}$, where Δ^n is the standard n-simplex.
(b) Show that if $\widetilde{\sigma}_1, \widetilde{\sigma}_2 : \Delta^n \to \widetilde{X}$ are two lifts of σ, then there is an element g of the fundamental group of X such that $g \circ \widetilde{\sigma}_1 = \widetilde{\sigma}_2$ where we view g as an automorphism of $\widetilde{X}$ via the deck transformations.

Hint: Δ^n simply connected. Suffices to find g so that it agrees at a point. True since deck transformations act transitively.

This is exactly parts (b) and (d) of Fall 2019-10.

Fall 2017

Fall 2017-1. Let M be a smooth manifold. Verify the following identity for vector fields X, Y and a 1-form ω on M:

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]).$$

Hint: Use local coordinates to assume $\omega = f dx$. Product rule for vector fields $X(fg) = gX(f) + fX(g)$.

We use local coordinates so $\omega = f_1 dx^1 + \cdots + f_n dx^n$ for some $f_i \in C^\infty(U)$ and $(U, x^1, \ldots, x^n)$ a local chart on M. By linearity of the differential operator, vector fields, and differential forms, it suffices to show the identity for $\omega = f_j dx^j$. Then, by the product rule we have

$$d\omega = d(f_j dx^j) = df_j \wedge dx^j + (-1)^0 f_j \wedge d(dx^j) = df_j \wedge dx^j.$$

Thus,

$$d\omega(X,Y) = (df_j \wedge dx^j)(X,Y) = df_j(X)dx^j(Y) - df_j(Y)dx^j(X) = X(f_j)Y(x^j) - X(x^j)Y(f_j).$$

On the other hand, $\omega(Y) = f_j dx^j(Y) = f_j Y(x^j)$ and similarly, $\omega(X) = f_j X(x^j)$ so, by the product rule,

$$\begin{aligned} X(\omega(Y)) - Y(\omega(X)) &= X(f_j Y(x^j)) - Y(f_j X(x^j)) \\ &= X(f_j)Y(x^j) + f_j XY(x^j) - (Y(f_j)X(x^j) + f_j YX(x^j)) \\ &= X(f_j)Y(x^j) - Y(f_j)X(x^j) + f_j[X,Y](x^j) \\ &= df_j \wedge dx^j(X,Y) + f_j dx^j([X,Y]) \\ &= d\omega(X,Y) + \omega([X,Y]), \end{aligned}$$

giving us the desired identity.

Fall 2017-2. Let $M_n(\mathbb{R})$ be the space of all $n \times n$ matrices with real coefficients.

(a) Show that $O(n) = \{A \in M_n(\mathbb{R}) \mid AA^T = \mathrm{id}\}$ is a smooth submanifold of $M_n(\mathbb{R})$. Here A^T is the transpose of A.
(b) Show that $O(n)$ has trivial tangent bundle.

Hint: $F : \mathrm{Mat}_{n\times n}(\mathbb{R}) \to \mathrm{Sym}_{n\times n}(\mathbb{R})$ by $M \mapsto M^T M$. Then $dF_A(\frac{1}{2}AC) = C$. Show that all Lie groups are parallelizable by finding a global frame. Take a basis $v_1, \ldots, v_n$ for T_eG and define $X_i(g) = dL_g(v_i)$.

This is exactly Fall 2022-2.

Fall 2017-3. The Hopf fibration $\pi : S^3 \to S^2$ is defined as follows: if we identify

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\},$$

and $S^2 = \mathbb{CP}^1$ with homogeneous coordinates $[z_1 : z_2]$, then $\pi(z_1, z_2) = [z_1 : z_2]$. There is another fibration $p : \mathbb{S}TS^2 \to S^2$, called the *unit tangent bundle*, whose fiber over $x \in S^2$ consist of the tangent vectors in T_xS^2 of unit length (here we may measure the length of a tangent vector by viewing S^2 as a submanifold of $\mathbb{R}^3$). Show that there is a covering map $f : S^3 \to \mathbb{S}TS^2$ of degree 2 satisfying $p \circ f = \pi$.

Hint: $\mathbb{S}TS^2 \cong SO(3) \cong \mathbb{RP}^3$. Standard $S^n \to \mathbb{RP}^n$ double cover by antipodal identification.

First, we note that $\mathbb{S}TS^2$ is diffeomorphic to $SO(3) \cong \mathbb{RP}^3$. To see this, fix notation

$$\mathbb{S}TS^2 = \{(p, v) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid |p| = |v| = 1, p \perp v\},$$

$$SO(3) = \{(v_1, v_2, v_3) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \mid (v_1, v_2, v_3) \text{ is an oriented orthonormal basis}\}.$$

Then, a diffeomorphism between $\mathbb{S}TS^2$ and $SO(3)$ is given by sending $(p, v) \in \mathbb{S}TS^2$ to $(p, v, p \times v)$ where the third vector is the cross product. Then, we know for any n, that $f : S^n \to \mathbb{RP}^n$ given by identifying antipodal points is a double cover of $\mathbb{RP}^n$. In particular, $f : S^3 \to \mathbb{RP}^3$ is the desired degree 2 covering map satisfying $p \circ f = \pi$.

Fall 2017-4. Consider the differential 1-form $\omega = xdy - ydx + dz$ in $\mathbb{R}^3$ with coordinates (x, y, z). Prove that $f\omega$ is not closed for any nowhere zero function $f : \mathbb{R}^3 \to \mathbb{R}$.

Hint: Just expand $d(f\omega) = df \wedge \omega + fd\omega$.

This is the same as Spring 2020-3 with an inconsequential sign change.

Fall 2017-5. Let x, y, z denote the standard Euclidean coordinates on $\mathbb{R}^3$ and let dA denote the standard area form on $S^2 = \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$. Determine the values of $n = 0, 1, 2, \ldots$ for which $\omega = z^n dA$ is an exact 2-form on S^2.

Hint: Exactly when n is odd. $dA = i^*\iota_N dV$ and $\iota_N dV = xdy \wedge dz - ydx \wedge dz + zdx \wedge dy$ so $d(z^n\iota_N dV) = (n+3)z^n dV$ and ω is exact if and only if $\int_{S^2} i^* z^n \iota_N dV = \int_B (n+3)z^n dV = 0$ by Stokes'.

Recall the definition of dA being induced by the volume form $dV = dx \wedge dy \wedge dz$ on $\mathbb{R}^3$: namely for $i : S^2 \hookrightarrow \mathbb{R}^3$ the inclusion, we have $dA = i^*\iota_N dV$ where N is the unit normal vector to S^2. So $\omega = z^n i^* \iota_N dV$. In our case, $N_p = p$ so we can explicitly write

$$N = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}.$$

We then note by linearity of ι_N and properties of the wedge product that

$$\begin{aligned}\iota_N dV &= \iota_{x\frac{\partial}{\partial x}}(dx \wedge dy \wedge dz) + \iota_{y\frac{\partial}{\partial y}}(dx \wedge dy \wedge dz) + \iota_{z\frac{\partial}{\partial z}}(dx \wedge dy \wedge dz)\\ &= xdy \wedge dz - ydx \wedge dz + zdx \wedge dy.\end{aligned}$$

Since ω is a top form on S^2, it is exact if and only if $\int_{S^2}\omega = 0$. By Stokes' theorem, we have

$$\int_{S^2}\omega = \int_{S^2} i^*(z^n \iota_N dV) = \int_B d(z^n \iota_N dV),$$

where B is the unit ball in $\mathbb{R}^3$. We compute

$$\begin{aligned}d(z^n\iota_N dV) &= d(z^n) \wedge \iota_N dV + (-1)^0 z^n \wedge d\iota_N dV\\ &= nz^{n-1}dz \wedge \iota_N dV + z^n d\iota_N dV\\ &= nz^{n-1}zdz \wedge dx \wedge dy + z^n(dx \wedge dy \wedge dz - dy \wedge dx \wedge dz + dz \wedge dx \wedge dy)\\ &= (nz^n + 3z^n)dx \wedge dy \wedge dz\\ &= (n+3)z^n dV,\end{aligned}$$

and hence $\int_{S^2}\omega = \int_B (n+3)z^n dV$. So ω is exact if and only if $\int_B(n+3)z^n dV = 0$. But we can see that this occurs exactly when n is odd since then z^n is an odd function in terms of z so the integral over the top $(z > 0)$ and bottom $(z < 0)$ half of the unit ball cancel each other out, while for n even, the integrand is always positive. Hence, ω is exactly if and only if n is odd.

Fall 2017-6. (a) Define what it means for a manifold M to be orientable. (You can give any one of the many equivalent definitions.)
(b) Show that every nonorientable connected manifold M admits a connected, oriented double cover.

Hint: $\widetilde{M} = \{(p, \mathcal{O}_p) \mid p \in M \text{ and } \mathcal{O}_p \text{ is an orientation of } T_pM\}$ with basis for topology $\widetilde{U}_{\mathcal{O}} = \{(p, \mathcal{O}_p) \mid p \in U, \mathcal{O}_p \text{ orientation of } T_pM \text{ determined by } \mathcal{O}\}$.

(a) An n-manifold M is orientable if it admits an atlas with each transition function xy^{-1} being an orientation preserving map between open subsets of $\mathbb{R}^n$. For maps $f : U \to V$ between open subsets of $\mathbb{R}^n$, we say f is orientation preserving if $\det(df_p) > 0$ at each point $p \in U$.

(b) This is exactly Fall 2021-7. The fact that $\widetilde{M}$ is connected if M is not orientable follows from noticing that each connected component of $\widetilde{M}$ is also a cover of M and is an oriented manifold. So if $\widetilde{M}$ is disconnected, it consists of two disjoint copies of M, both of which are orientable, contradicting the fact that M is not orientable.

Fall 2017- 7. Let M be a smooth, compact, connected oriented n-dimensional manifold (without boundary).

(a) Show that if the Euler characteristic of M is zero, then M admits a nowhere vanishing vector field.
(b) If M is a surface of genus g, then what is $\min_v(\#\text{ zeros of } v)$, where v ranges over vector fields on M whose zeros are isolated and have index ± 1? Give a proof.

Hint: Sum of indices of isolated zeros is zero. Put them all in a neighborhood diffeomorphic to a ball and the function on the boundary $X_p/||X_p||$ has degree zero so can be extended into the ball. Define new vector field based on this. Minimum is $2g - 2 = -\chi(M)$. Do "dunking" vector field, then similarly move the zeros around so they cancel.

Referenced in: Fall 2016-6, Spring 2009-3.

(a) First we may find a vector field X on M with a finite number of isolated zeros since M is compact. Consider an open chart (V, ϕ) diffeomorphic to $\mathbb{R}^n$. Let $U = \phi^{-1}(B_1(0))$. Without loss of generality, we may assume all of these zeros are located in U. By the Poincaré-Hopf index theorem, the sum of the indices of these zeros is equal to the Euler characteristic which is zero. Let $f : \partial\overline{U} \to S^{n-1}$ be defined by $p \mapsto \frac{X_p}{||X_p||}$ where we treat $\overline{U} \cong \overline{B_1(0)}$. Then, the sum of these indices is exactly the degree of this map so $\deg(f) = 0$.

Then, by the extension theorem, we may extend f to a function g defined on all of $\overline{U}$. Define a vector field Y on M by

$$Y_p = \begin{cases} g(p) & p \in \overline{U} \\ \frac{X_p}{||X_p||} & p \notin U, \end{cases}$$

which is well-defined since $X_p \neq 0$ for $p \notin U$ and the two cases agree on the boundary of U. Treating $Y|_V$ as a function $V \to S^{n-1}$, by the Whitney approximation theorem, we may make $Y|_V$ smooth in a way that doesn't change $Y|_{V-\phi^{-1}(B_{1.5}(0))}$ (because the only place where Y may not be smooth is at $\partial\overline{U}$). Now, Y is nowhere vanishing since X is nonvanishing outside of U and $g(p) \in S^{n-1}$ is never zero either.

(b) We know that $\chi(M) = 2 - 2g$ so the sum of the indices of the zeros of X is $2 - 2g$. Since each zero has index ± 1, it is clear that the minimum number of zeros would be $2 - 2g$, each having index -1. We can attain this minimum as follows. Start with a vector field X that has one source, one sink, and $2g$ saddles which can be constructed by considering the flow of "dunking an n-hole donut and lifting it".

Without loss of generality, assume that the source, sink, and two of the saddles are contained in a neighborhood U diffeomorphic to the unit ball $B_1(0) \subset \mathbb{R}^2$. Again, $f(p) = \frac{X_p}{||X_p||}$ is nonzero on $\partial\overline{U} \cong S^1$ and has degree 0 since the sum of the indices of the source, sink, and two saddles is 0. So we may extend f to a map g on all of $\overline{U}$. Now, however, X_p has zeros in $M - \overline{U}$. To account for this, on successively smaller opens, transition smoothly from X_p to $\frac{X_p}{||X_p||}$ to $g(p)$ and let this vector field be Y. This Y is the desired vector field with only $2g - 2$ zeros, namely those outside of $M - \overline{U}$.

Fall 2017-8. Let $M = [0,1] \times [0,1]/\sim_0$, where $(x,1) \sim_0 (1-x,0)$ for all $x \in [0,1]$, and let $X = (M \times \{0,1\})/\sim_1$, where $(y,1) \sim_1 (y,0)$ for all $y \in \partial M$. Determine the fundamental group of X.

Hint: Klein bottle. $\pi_1(X) = \langle a, b \mid bab^{-1}a \rangle \cong \mathbb{Z} \rtimes \mathbb{Z}$. Polygon representation, CW structure.

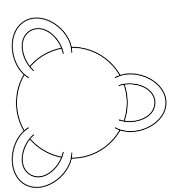
Referenced in: Fall 2014-8.

We note that M is the Möbius band and X is the Klein-bottle. Now, we know that a polygon representation of X is given by $[0,1]^2/\sim$ where $(x,0) \sim (x,1)$ and $(0,y) \sim (1,1-y)$ for all $x, y \in [0,1]$. This then has a CW structure given by two 1-cells, a and b, where a corresponds to the path from $(1,0)$ to $(1,1)$ and b corresponds to the path from $(0,0)$ to $(1,0)$, and one 2-cell attached via $bab^{-1}a$. Hence, the fundamental group is

$$\pi_1(X) = \langle a, b \mid bab^{-1}a \rangle.$$

By inspection we see that this group presentation is isomorphic to $\mathbb{Z} \rtimes \mathbb{Z}$, the non-trivial semidirect product of the integers with itself.

Fall 2017-9. A compact surface (without boundary) of genus g, embedded in $\mathbb{R}^3$ in the standard way (see below of the case $g = 3$), bounds a compact 3-dimensional region called a *handlebody* H (the region "inside" the surface in the following figure).



Let $X = (H \times \{0,1,2\})/\sim$, where $(x,i) \sim (x,j)$ for all $x \in \partial H$ and $i, j \in \{0,1,2\}$. Compute the homology of X.

Hint: $H_*(X) = \mathbb{Z}^2_{(3)} \oplus \mathbb{Z}^{2g}_{(2)} \oplus \mathbb{Z}^g_{(1)} \oplus \mathbb{Z}_{(0)}$. Good pair (X, H) where we know $H_*(X, H) = (H_*(H, \partial H))^2$ and we compute $H_*(H, \partial H)$ using Lefschetz duality/universal coefficient formula from $H_k(H)$. The important map $\mathbb{Z}^{2g} \to \mathbb{Z}^g$ is zero. It is also possible to use Mayer-Vietoris with neighborhood of H.

Referenced in: Fall 2014-7.

First, we see that H deformation retracts onto the wedge sum of g circles so we have

$$H_k(H) = \begin{cases} \mathbb{Z}^g & k = 1, \\ \mathbb{Z} & k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

We can easily compute the cohomology of H using the universal coefficient formula since all of the homology groups are free. Then, since H is a compact orientable 3-manifold, we may use Lefschetz duality to get

$$H_k(H, \partial H) = H^{3-k}(H) = H_{3-k}(H) = \begin{cases} \mathbb{Z}^g & k = 2, \\ \mathbb{Z} & k = 3, \\ 0 & \text{otherwise.} \end{cases}$$

Since $(H, \partial H)$ is a good pair, so too is (X, H). In particular, note that X/H is the wedge sum of two copies of $H/\partial H$ so

$$H_k(X, H) = \widetilde{H}_k(X/H) = \widetilde{H}_k(H/\partial H) \oplus \widetilde{H}_k(H/\partial H) = H_k(H, \partial H) \oplus H_k(H, \partial H) = \begin{cases} \mathbb{Z}^{2g} & k = 2, \\ \mathbb{Z}^2 & k = 3, \\ 0 & \text{otherwise.} \end{cases}$$

Then, the long exact sequence for the pair (X, H) is

$$\cdots \to \widetilde{H}_k(H) \to \widetilde{H}_k(X) \to \widetilde{H}_k(X, H) \to \widetilde{H}_{k-1}(H) \to \cdots,$$

which using the above simplifies to

$$\cdots \to 0 \to \widetilde{H}_3(X) \to \mathbb{Z}^2 \to 0 \to \widetilde{H}_2(X) \to \mathbb{Z}^{2g} \xrightarrow{f} \mathbb{Z}^g \to \widetilde{H}_1(X) \to 0 \to \cdots,$$

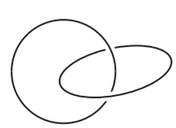
where we have already computed $\widetilde{H}_0(X) = 0$ since X is clearly connected. The first part of the sequence gives us $\widetilde{H}_3(X) \cong \mathbb{Z}^2$ while the second part gives $\widetilde{H}_2(X) = \ker(f)$ and $\widetilde{H}_1(X) = \operatorname{coker}(f)$. However, we can see that the induced boundary map f must actually be 0 so $\widetilde{H}_2(X) \cong \mathbb{Z}^{2g}$ and $\widetilde{H}_1(X) \cong \mathbb{Z}^g$. Hence, to summarize, we have

$$H_k(X) = \begin{cases} \mathbb{Z} & k = 0, \\ \mathbb{Z}^g & k = 1, \\ \mathbb{Z}^{2g} & k = 2, \\ \mathbb{Z}^2 & k = 3, \\ 0 & \text{otherwise.} \end{cases}$$

Fall 2017-10. (a) Let A be a single circle in $\mathbb{R}^3$. Compute $\pi_1(\mathbb{R}^3 - A, *)$.

(b) Let A and B be disjoint circles in $\mathbb{R}^3$, supported in the upper and lower half space, respectively. Compute $\pi_1(\mathbb{R}^3 - (A \cup B), *)$.

(c) How does $\pi_1(\mathbb{R}^3 - (A \cup B), *)$ change if the circles are linked as below?



Hint: Homotopy equivalent to $S^1 \vee S^2, S^1 \vee S^1 \vee S^2$, and $T^2 \vee S^2$ respectively. Or use Van Kampen's.

Referenced in: Fall 2013-9.

(a) Note that $\mathbb{R}^3 - A$ is homotopy equivalent to $S^1 \vee S^2$. So by Van Kampen's theorem, since $S^1 \cap S^2 = \{*\}$ is contractible,

$$\pi_1(\mathbb{R}^3 - A) \cong \pi_1(S^1 \vee S^2) = \pi_1(S^1) * \pi_1(S^2) = \mathbb{Z} * 1 = \mathbb{Z}.$$

(b) Now, $\mathbb{R}^3 - (A \cup B)$ is homotopy equivalent to the wedge sum of two copies of $\mathbb{R}^3 - A$ so by Van Kampen's,

$$\pi_1(\mathbb{R}^3 - (A \cup B)) \cong \pi_1((\mathbb{R}^3 - A) \vee (\mathbb{R}^3 - B)) = \pi_1(\mathbb{R}^3 - A) * \pi_1(\mathbb{R}^3 - B) = \mathbb{Z} * \mathbb{Z}.$$

(c) In this case, $\mathbb{R}^3 - (A \cup B)$ is homotopy equivalent to the wedge sum of S^2 and T^2, the standard torus. Again, by Van Kampen's, we get

$$\pi_1(\mathbb{R}^3 - (A \cup B)) = \pi_1(T^2 \vee S^2) = \pi_1(T^2) * \pi_1(S^2) = \mathbb{Z}^2 * 1 = \mathbb{Z}^2.$$

Another way to see this is by first noting that we can add a point at infinity to $\mathbb{R}^3$ to get $\mathbb{R}^3 \subset S^3 = \mathbb{R}^3 \cup \{\infty\}$. Doing this does not change the fundamental group by Van Kampen's theorem since we can find a neighborhood of $A \cup B$ in S^3 that avoids ∞ (since $A \cup B$ is bounded) and a neighborhood of ∞ homeomorphic to an open ball that together cover S^3. Then, removing any point from S^3 gives $\mathbb{R}^3$ so consider first removing a point p on the circle A.

Thus $S^3 - (A \cup B)$ is equal to $S^3 - \{p\} - (A \cup B - \{p\}) \simeq \mathbb{R}^3 - (L \cup B)$ where L is a line through the center of B (think of uncurling the circle A once we remove the point p). But then, this more clearly deformation retracts onto the standard torus in $\mathbb{R}^3$ so $\pi_1(\mathbb{R}^3 - (A \cup B)) = \pi_1(S^3 - (A \cup B)) = \pi_1(T^2) = \mathbb{Z}^2$ as claimed above.

Spring 2017

Spring 2017-1. Let M be a connected smooth manifold of dimension at least two. Prove that for any $2n$ distinct points

$$x_1, \ldots, x_n, y_1, \ldots, y_n \in M$$

there exists a diffeomorphism $f : M \to M$ such that $f(x_i) = y_i$ for $i = 1, \ldots, n$.

Hint: Do each $x_i \mapsto y_i$ separately and compose. Need to use connectedness. Compact support for each.

This is just an easier version of Fall 2020-1.

Spring 2017-2. Let $M_{2n\times 2n}(\mathbb{R}) = \mathbb{R}^{4n^2}$ be the space of $2n \times 2n$ real matrices. Consider the following matrix in block form

$$\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \in M_{2n\times 2n}(\mathbb{R}),$$

where I_n is the $n \times n$ identity matrix. Show that the subspace

$$S = \{A \in M_{2n\times 2n}(\mathbb{R}) \mid A^T \Omega A = \Omega\}$$

is a smooth submanifold of $M_{2n\times 2n}(\mathbb{R})$, and compute its dimension. (Here A^T denotes the transpose of A.)

Hint: Preimage theorem $F : M_{2n}(\mathbb{R}) \to Sk_{2n}(\mathbb{R})$ of skew symmetric matrices by $A \mapsto A^T \Omega A$. For $A \in F^{-1}(\Omega)$, $C \in Sk_{2n}(\mathbb{R})$, take $B = \frac{1}{2}\Omega^{-1}(A^{-1})^T C$ so that $dF_A(B) = C$.

Let

$$Sk_{2n}(\mathbb{R}) = \{A \in M_{2n}(\mathbb{R}) \mid A^T = -A\}$$

be the set of skew symmetric $2n \times 2n$ matrices. It is clear that $Sk_{2n}(\mathbb{R}) \subset M_{2n}(\mathbb{R})$ is a smoothly embedded submanifold. It has dimension $\binom{2n}{2}$ since a matrix in $Sk_{2n}(\mathbb{R})$ is uniquely determined by the its (i, j) entries for $i < j$ and the diagonal entries must be zero. Now, define

$$F : M_{2n}(\mathbb{R}) \to Sk_{2n}(\mathbb{R}), \quad A \mapsto A^T \Omega A.$$

This is well-defined since $(A^T\Omega A)^T = A^T\Omega^T A = -A^T\Omega A$ so the output will always indeed be skew symmetric. We claim that Ω is a regular value of F. We can compute the differential of F at $A \in M_{2n}(\mathbb{R})$ to be

$$\begin{aligned} dF_A(B) = \lim_{t\to 0} \frac{F(A + tB) - F(A)}{t} &= \lim_{t=0} \frac{(A + tB)^T\Omega(A + tB) - A^T\Omega A}{t} \\ &= \lim_{t\to 0} \frac{A^T\Omega A + tB^T\Omega A + tA^T\Omega B + t^2B^T\Omega B - A^T\Omega A}{t} \\ &= \lim_{t\to 0} B^T\Omega A + A^T\Omega B + tB^T\Omega B \\ &= B^T\Omega A + A^T\Omega B. \end{aligned}$$

Now, if $A \in F^{-1}(\Omega)$, then $A^T\Omega A = \Omega$, implying that $\det(A^T\Omega A) = \det(\Omega)$ so $\det(A)^2 = 1$ since $\det(A^T) = \det(A)$ and $\det(\Omega) = (-1)^n \neq 0$. Hence, any $A \in F^{-1}(\Omega)$ is invertible and so too is Ω. So take $A \in F^{-1}(\Omega)$ and let $C \in Sk_{2n}(\mathbb{R}) \cong T_\Omega(Sk_{2n}(\mathbb{R}))$. Choose $B = \frac{1}{2}\Omega^{-1}(A^{-1})^T C$ so that

$$B^T\Omega A = (A^T\Omega^T B)^T = -(A^T\Omega B)^T = -\frac{1}{2}C^T = \frac{1}{2}C \text{ and } A^T\Omega B = \frac{1}{2}C.$$

Thus, we have $dF_A(B) = B^T\Omega A + A^T\Omega B = \frac{1}{2}C + \frac{1}{2}C = C$ so

$$dF_A : T_A(M_{2n}(\mathbb{R})) \to T_\Omega(Sk_{2n}(\mathbb{R}))$$

is surjective for any $A \in F^{-1}(\Omega)$ showing that Ω is a regular value of F. Hence, by the preimage theorem, $F^{-1}(\Omega) = S$ is a smooth submanifold of $M_{2n}(\mathbb{R})$ of codimension equal to the dimension of $Sk_{2n}(\mathbb{R})$. I.e., S has dimension $4n^2 - \binom{2n}{2} = 4n^2 - n(2n - 1) = 2n^2 + n$.

Spring 2017-3. Use the Poincaré-Hopf index theorem to calculate the Euler characteristic of the n-dimensional sphere S^n. (You must compute the indices in local coordinates. Drawings do not suffice!)

Hint: $\chi(S^n) = 0$ if n is odd and 2 if n is even. Standard nowhere vanishing vector field on S^n for n odd and for n even, do the above for first $n - 1$ coordinates and set last one to zero. Two zeros, each of index 1.

For $n = 2k - 1$ odd, we have a nowhere vanishing vector field X, on $S^n \subset \mathbb{R}^{2k}$ given by

$$X : (x_1, x_2, x_3, x_4, \ldots, x_{2k-1}, x_{2k}) \mapsto (-x_2, x_1, -x_4, x_3, \ldots, -x_{2k}, x_{2k-1})$$

since $x \cdot X(x) = -x_1x_2 + x_2x_1 - x_3x_4 + x_4x_3 - \cdots - x_{2k-1}x_{2k} + x_{2k}x_{2k-1} = 0$ for any $x \in S^n$. Hence, the Euler characteristic must be 0 in this case, $\chi(S^n) = 0$ by the Poincaré-Hopf index theorem as we sum over no zeros.

For $n = 2k$ even, we consider the vector field X on $S^n \subset \mathbb{R}^{2k+1}$ given by

$$X : (x_1, x_2, x_3, x_4, \ldots, x_{2k-1}, x_{2k}, x_{2k+1}) \mapsto (-x_2, x_1, -x_4, x_3, \ldots, -x_{2k}, x_{2k-1}, 0).$$

As above, this is a vector field on S^n since $x \cdot X(x) = 0$ for any $x \in S^n$. Now, $X(x) = 0$ if and only if $x_1 = \ldots = x_{2k} = 0$ which gives two isolated zeros on S^n, namely $x_\pm := (0, \ldots, 0, \pm 1)$. By Poincaré-Hopf, we thus have $\chi(S^n) = \operatorname{ind}_{x_+}(X) + \operatorname{ind}_{x_-}(X)$.

To compute the first index, let $B = \{x \in S^n \mid x_{2k+1} \geqslant 0\}$ which contains x_+ but not x_-. Then the index of X at x_+ is the degree of the map $f : \partial B \to S^{n-1}$ given by $y \mapsto \frac{X_y}{||X_y||}$. With this choice, we have $\partial B = \{x \in S^n \mid x_{2k+1} = 0\} = S^{n-1}$ is exactly the equatorial S^{n-1} in S^n. Observe that f is defined as follows

$$(x_1, \ldots, x_{2k}, 0) \in S^{n-1} \mapsto \frac{X_{(x_1,\ldots,x_{2k},0)}}{||X_{(x_1,\ldots,x_{2k},0)}||} = \frac{(-x_2, \ldots, x_{2k-1}, 0)}{||(-x_2, \ldots, x_{2k-1}, 0)||} = (-x_2, \ldots, x_{2k-1}, 0) \in S^{n-1}.$$

I.e., f is the composition of k flips and k coordinate-negations so we know that $\deg(f) = (-1)^k \cdot (-1)^k = 1$. Similarly, to compute the index of x_-, we can take $B' = \{x \in S^n \mid x_{2k+1} \leqslant 0\}$. We again get $\partial B' = \partial B = S^{n-1}$ and the map is exactly the same so the degree is also 1. Hence, $\chi(S^n) = 1 + 1 = 2$ when n is even.

Spring 2017-4. (a) State the Cartan formula (also known as Cartan's magic formula) for the Lie derivative of a differential form with respect to a vector field.

(b) Use this formula to show that a vector field X on $\mathbb{R}^3$ has a flow (defined locally and for a short time) that preserves volume if and only if the divergence of X is everywhere zero. (Here, the divergence is the classical operator that takes a vector field with components f, g, h to the function $\frac{\partial}{\partial x} f + \frac{\partial}{\partial y} g + \frac{\partial}{\partial z} h$, where x, y, z are the usual coordinates on $\mathbb{R}^3$.)

Hint: ϕ preserves volume if and only if $\phi_t^* \omega = \omega$ for small t where $\omega = dx \wedge dy \wedge dz$ is the standard volume form on $\mathbb{R}^3$. Show this if and only if $L_X \omega = d\iota_X \omega = 0$.

Referenced in: Spring 2011-2.

(a) Let X be a vector field on a smooth manifold M and let ω be a k-form on M. Then $L_X \omega = d\iota_X \omega + \iota_X d\omega$ where d is the exterior derivative and ι_X is the interior product defined by

$$\iota_X \omega(X_1, \ldots, X_{k-1}) = \omega(X, X_1, \ldots, X_{k-1}).$$

(b) Note that a flow ϕ preserves volume if and only if each pullback of the volume form $\omega = dx \wedge dy \wedge dz$ is equal to the volume form itself, i.e., $\phi_t^* \omega = \omega$ for small t. Using the limit definition of L_X, we then know that $L_X(\omega) = 0$. Conversely, if $L_X(\omega) = 0$, then $\phi_{t_0}^* L_X(\omega) = 0$ so $L_X(\phi_{t_0}^* \omega) = 0$ where the commutativity here comes from continuity of ϕ.

By definition, we have $L_X(\phi_{t_0}^* \omega) = \frac{d}{dt} |_{t=t_0} \phi_t^* \omega$ so since t_0 was arbitrary, we get that $\phi_t^* \omega$ is constant. Then, $\phi_t^* \omega = \phi_0^* \omega = \omega$ for small t. This holds for each point p (which we should keep fixed) and so works overall point-wise. Hence, X preserves volume if and only if $L_X \omega = 0$. By Cartan's magic formula, $L_X \omega = d\iota_X \omega + \iota_X d\omega = d\iota_X \omega$ because $d\omega = 0$ since ω is a top form.

Let $X = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} + h\frac{\partial}{\partial z}$ so using the linearity and definition of ι, ω we have

$$\begin{aligned} d\iota_X \omega &= d(X(x) dy \wedge dz - X(y) dx \wedge dz + X(z) dx \wedge dy) \\ &= d(f dy \wedge dz - g dx \wedge dz + h dx \wedge dy) \\ &= (\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}) dx \wedge dy \wedge dz \\ &= (\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}) \omega. \end{aligned}$$

So X preserves volume if and only if $(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z})\omega = 0$ but ω is nowhere vanishing to this occurs if and only if $(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}) = 0$ as desired.

Spring 2017-5. Let

$$\omega = \frac{-ydx + xdy}{(x^2+y^2)^\alpha}$$

be a 1-form on $\mathbb{R}^2 - \{0\}$ with the usual coordinates (x, y), and for some $\alpha \in \mathbb{R}$. Consider $\int_\gamma \omega$, where $\gamma : S^1 \to \mathbb{R}^2 - \{0\}$ is a smooth map.

(a) For which $\alpha \in \mathbb{R}$ do we have $\int_{\gamma_0} \omega = \int_{\gamma_1} \omega$ whenever γ_0 and γ_1 are *smoothly homotopic* (i.e., there exists a smooth map $F : S^1 \times [0,1] \to \mathbb{R}^2 - \{0\}$ such that $\gamma_0(t) = F(t,0), \gamma_1(t) = F(t,1)$)?

(b) What are the possible values of $\int_\gamma \omega$ when α is chosen as in part (a)?

Hint: ω closed implies integrals are the same. Integrals being the same implies that $\int_{S^1(R)} \omega$ is independent of R. Convert to polar coordinates and solve. $\alpha = 1$. (b) $2\pi k$ for $k \in \mathbb{Z}$ since every loop homotopic to $k \cdot S^1$.

(a) First, suppose that ω is closed. Then, let $F : S^1 \times [0,1] \to \mathbb{R}^2 - \{0\}$ be a smooth homotopy between the loops γ_0 and γ_1. Then, since ω is closed, letting $K = S^1 \times [0,1]$, we have

$$\int_K d(F^*\omega) = \int_K F^* d\omega = 0.$$

By Stokes' theorem, we also have

$$\int_K d(F^*\omega) = \int_{\partial K} F^*\omega = \int_{S^1\times\{0\}} F^*\omega - \int_{S^1\times\{1\}} F^*\omega = \int_{\gamma_0} \omega - \int_{\gamma_1} \omega.$$

Hence we must have $\int_{\gamma_0} \omega = \int_{\gamma_1} \omega$. Conversely, suppose that $\int_{\gamma_0} \omega = \int_{\gamma_1} \omega$ for any two γ_0, γ_1 smoothly homotopic. In particular, note that the circle of radius R which we will denote $S^1(R) \subset \mathbb{R}^2 - \{0\}$ and the unit circle $S^1 \subset \mathbb{R}^2 - \{0\}$ are smoothly homotopic. Thus,

$$\int_{S^1(R)} \omega = \int_{S^1} \omega,$$

implying that $\int_{S^1(R)} \omega$ is independent of $R > 0$. Using polar coordinates $x = r\cos(\theta), y = r\sin(\theta)$, we have

$$\begin{aligned}
\omega &= \frac{-r\sin(\theta)(\cos(\theta)dr - sin(\theta)rd\theta) + r\cos(\theta)(\sin(\theta)dr + \cos(\theta)rd\theta)}{r^{2\alpha}} \\
&= \frac{-r\sin(\theta)\cos(\theta)dr + r^2\sin^2(\theta)d\theta + r\cos(\theta)\sin(\theta)dr + r^2\cos^2(\theta)d\theta}{r^{2\alpha}} \\
&= \frac{r^2 d\theta}{r^{2\alpha}} = r^{2-2\alpha} d\theta.
\end{aligned}$$

Thus,

$$\int_{S^1(R)} \omega = \int_0^{2\pi} R^{2-2\alpha} d\theta = 2\pi R^{2-2\alpha}.$$

For this to be independent of R, we must have $\alpha = 1$ so this is our only possibility. Letting $\omega = \frac{-ydx+xdy}{x^2+y^2}$, we can compute

$$\begin{aligned}
d\omega &= -d\left(\frac{y}{x^2+y^2}dx\right) + d\left(\frac{x}{x^2+y^2}dy\right) \\
&= -\frac{(x^2+y^2)dy - y(2xdx + 2ydy)}{(x^2+y^2)^2} \wedge dx + \frac{(x^2+y^2)dx - x(2xdx+2ydy)}{(x^2+y^2)^2} \wedge dy \\
&= \frac{x^2-y^2}{(x^2+y^2)^2} dx \wedge dy + \frac{y^2-x^2}{(x^2+y^2)^2} dx \wedge dy = 0,
\end{aligned}$$

so ω is closed. Hence $\int_{\gamma_0} \omega = \int_{\gamma_1} \omega$ for all γ_0 and γ_1 smoothly homotopic if and only if $\alpha = 1$.

(b) Since $\mathbb{R}^2 - \{0\}$ deformation retracts onto $S^1 \subset \mathbb{R}^2 - \{0\}$, any loop in $\mathbb{R}^2 - \{0\}$ is smoothly homotopic to a loop $k \cdot S^1$ that goes counterclockwise k times around S^1 for some $k \in \mathbb{Z}$. Then, using the calculations above, we have

$$\int_\gamma \omega = \int_{k \cdot S^1} \omega = k \int_{S^1} \omega = k \int_0^{2\pi} d\theta = 2\pi k$$

for $k \in \mathbb{Z}$.

Spring 2017-6. Let X and Y be connected CW-complexes, let $p : \widetilde{X} \to X$ be a path connected covering space, and let $f : Y \to X$ be an arbitrary continuous map. Let

$$f^*(\widetilde{X}) = \{(y, \widetilde{x}) \mid f(y) = p(\widetilde{x})\} \subset Y \times \widetilde{X},$$

and consider the projection map $f^*(p) : f^*(\widetilde{X}) \to Y$, $f^*(p)(y, \widetilde{x}) = y$.

(a) Show that $f^*(p)$ is a covering map.

(b) Let $(y, \widetilde{x}) \in f^*(\widetilde{X})$, and let $x = f(y) = p(\widetilde{x})$. If

$$f_* \pi_1(Y, y) \subset p_* \pi_1(\widetilde{X}, \widetilde{x}),$$

and the cover $p : \widetilde{X} \to X$ is nontrivial, show that $f^*(\widetilde{X})$ is disconnected.

Hint: Evenly cover $f(y)$. Define $W_\alpha = q^{-1}(U_\alpha)$ where $q((y, \widetilde{x})) = \widetilde{x}$. Then, $z \mapsto (z, p|_{U_\alpha}^{-1} f(z))$ is inverse to $\pi|_{W_\alpha}$. (b) Contradiction, consider path γ between (y, a) and (y, b) where a and b are both in the fiber over x. Show $q_* \gamma$ is homotopic to a loop in $\widetilde{X}$ so $a = b$.

Referenced in Spring 2009-8.

(a) We will denote $f^*(p)$ by π and define $q : f^*(\widetilde{X}) \to \widetilde{X}$ by $q((y, \widetilde{x})) = \widetilde{x}$. Note that $f \circ \pi((y, \widetilde{x})) = f(y) = p(\widetilde{x}) = p \circ q((y, \widetilde{x}))$ so we have the following commutative diagram:

$$\begin{array}{ccc} f^*(\widetilde{X}) & \xrightarrow{q} & \widetilde{X} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow[f]{} & X \end{array}$$

Let $y \in Y$ be arbitrary and let $x = f(y)$. Since $p : \widetilde{X} \to X$ is a covering map, let U be a neighborhood of x in X that is evenly covered by p, i.e., $p^{-1}(U) = \bigsqcup_\alpha U_\alpha$ and $p|_{U_\alpha} : U_\alpha \to U$ is a diffeomorphism for each α. Since $f\pi = pq$, we have

$$\pi^{-1}(f^{-1}(U)) = q^{-1}(p^{-1}(U)) = q^{-1}(\bigsqcup_\alpha U_\alpha) = \bigsqcup_\alpha q^{-1}(U_\alpha), \text{ so define } W_\alpha = q^{-1}(U_\alpha) \subset f^*(\widetilde{X}).$$

Then, $W = f^{-1}(U)$ is a neighborhood of y in Y and $\pi^{-1}(W) = \bigsqcup_\alpha W_\alpha$. Now, we want to show that $\pi|_{W_\alpha} : W_\alpha \to W$ is a diffeomorphism. To see this, we construct the inverse

$$\phi_\alpha : W \to W_\alpha, \quad z \mapsto (z, p|_{U_\alpha}^{-1} f(z)).$$

Since $z \in W$, we have $f(z) \in U$ so $p|_{U_\alpha}^{-1}(f(z)) \in U_\alpha$. But also,

$$f(z) = p(p|_{U_\alpha}^{-1} f(z)) \implies (z, p|_{U_\alpha}^{-1} f(z)) \in f^*(\widetilde{X}) \text{ and } q((z, p|_{U_\alpha}^{-1} f(z))) = p|_{U_\alpha}^{-1} f(z) \in U_\alpha,$$

which implies that

$$(z, p|_{U_\alpha}^{-1} f(z)) \in q^{-1}(U_\alpha) = W_\alpha,$$

so ϕ_α is well-defined. Now, it is also clear from this definition that ϕ_α is smooth and an inverse to $\pi|_{W_\alpha}$ so indeed $\pi|_{W_\alpha}$ is a diffeomorphism for each α showing that π is a covering map.

(b) Suppose for sake of contradiction that $f^*(X)$ is connected. Since p is nontrivial, we may find $a \neq b \in \widetilde{X}$ such that $p(a) = p(b) = x$. Thus, $(y, a), (y, b) \in f^*(\widetilde{X})$ both get sent to y by π. Let γ be a path from (y, a) to (y, b) since $f^*(X)$ is connected. Then, $\pi \circ \gamma$ is a path in Y that starts and ends at y, i.e., a loop based at y.

By assumption, $f_*(\pi \circ \gamma) \in f_*\pi_1(Y, y) \subset p_*\pi_1(\widetilde{X}, \widetilde{x})$ so $f_*(\pi \circ \gamma) = p_*\beta$ for some loop β in $\widetilde{X}$ based at $\widetilde{x}$. But, we know that

$$f_*(\pi \circ \gamma) = (f_* \circ \pi_*)(\gamma) = (f \circ \pi)_*(\gamma) = (p \circ q)_*(\gamma) = p_*(q \circ \gamma).$$

Hence $p_*(q \circ \gamma) = p_*\beta$. Then, $p_*(q \circ \gamma)$ lifts to the path $q \circ \gamma$ in $\widetilde{X}$ and $p_*\beta$ lifts to the loop β in $\widetilde{X}$. Thus, $q \circ \gamma$ is homotopic to β as paths in $\widetilde{X}$ so $q \circ \gamma$ is also a loop based at $\widetilde{x}$ which means that $q((y, a)) = a = \widetilde{x}$ and $q((y, b)) = b = \widetilde{x}$, contradicting the fact that $a \neq b$. Hence $f^*(X)$ must be disconnected.

Spring 2017-7. Let $X = S^1 \times D^2$ with boundary $\partial X = S^1 \times S^1$. Compute the relative homology groups $H_k(X, \partial X; \mathbb{Z})$ for all k.

Hint: $H_*(X, \partial X) = \mathbb{Z}_{(3)} \oplus \mathbb{Z}_{(2)}$. Good pair $(X, \partial X)$ with $\widetilde{H}_0(X/\partial X) = 0$.

First note that D^2 is contractible so $S^1 \times D^2$ is homotopy equivalent to S^1. Then, we know

$$H_k(X) = H_k(S^1) = \begin{cases} \mathbb{Z} & k = 0, 1, \\ 0 & \text{otherwise.} \end{cases}, \quad H_k(\partial X) = H_k(S^1 \times S^1) = \begin{cases} \mathbb{Z} & k = 0, 2, \\ \mathbb{Z}^2 & k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Also, (D^2, S^1) is a good pair so $(X, \partial X)$ is as well, giving us a long exact sequence

$$\cdots \to H_k(\partial X) \to H_k(X) \to H_k(X, \partial X) \to H_{k-1}(\partial X) \to \cdots,$$

which given the above groups becomes

$$\cdots \to 0 \to H_3(X, \partial X) \to \mathbb{Z} \to 0 \to H_2(X, \partial X) \to \mathbb{Z}^2 \xrightarrow{f} \mathbb{Z} \to H_1(X, \partial X) \xrightarrow{g} \mathbb{Z} \xrightarrow{h} \mathbb{Z} \to H_0(X, \partial X) \to 0.$$

In degree 3, we thus have $H_3(X, \partial X) \cong \mathbb{Z}$. In degree 0, we know $H_0(X, \partial X) \cong \widetilde{H}_0(X/\partial X) = 0$ since $X/\partial X$ is connected. Thus $g = 0$ since h is an isomorphism so $H_2(X, \partial X) = \ker(f)$ and $H_1(X, \partial X) = \operatorname{coker}(f)$. Now, f is induced by the inclusion $\partial X \hookrightarrow X$ which means that $f : (a, b) \mapsto a$ where a is a loop that wraps around the first S^1 and b around the second. I.e., $\ker(f) \cong \mathbb{Z}$ and $\operatorname{coker}(f) = 0$. To summarize, we have

$$H_k(X, \partial X) = \begin{cases} \mathbb{Z} & k = 2, 3, \\ 0 & \text{otherwise.} \end{cases}$$

Spring 2017-8. Let X be a CW complex and let $\widetilde{X} \to X$ be a covering space. let G be the group of deck transformations of $\widetilde{X} \to X$.

(a) Show that for any k and for any abelian group M, the group G acts naturally on $H_k(\widetilde{X}; M)$.
(b) Show that the map $p_* : H_k(\widetilde{X}; M) \to H_k(X; M)$ factors through the quotient of $H_k(\widetilde{X}; M)$ by the subgroup S generated by $m - g \cdot m$ for all $m \in H_k(\widetilde{X}; M)$ and $g \in G$.
(c) Give an example for which the induced map $H_k(\widetilde{X}; M)/S \to H_k(X; M)$ in (b) is not surjective.

Hint: Induced map by $g : \widetilde{X} \to \widetilde{X}$. $pg = p$. $p : \mathbb{R} \to S^1$, use universal coefficient theorem to get $H_1(S^1; M) \cong M$ while $H_1(\mathbb{R}; M) = 0$.

(a) We know that a deck transformation $g \in G$ acts on $\widetilde{X}$, i.e., $g : \widetilde{X} \to \widetilde{X}$ is a map. For any k and any abelian group M, $H_k(-; M)$ is a functor so g naturally induces a map $g_* : H_k(\widetilde{X}; M) \to H_k(\widetilde{X}; M)$ which is indeed an action on $H_k(\widetilde{X}; M)$.

(b) This is equivalent to showing that $p_*(s) = 0$ for all $s \in S$. For this, it suffices to show that $p_*(m - g \cdot m) = 0$ for all $m \in H_k(\widetilde{X}; M)$ and $g \in G$. Since p_* is an abelian group homomorphism, we have $p_*(m - g \cdot m) = p_*(m) - p_*(g \cdot m) = p_*(m) - p_* g_*(m)$ but we know that g sends elements in $\widetilde{X}$ to other elements in the same fiber over p. I.e., $p \circ g = p$ so $p_* g_* = p_*$ and hence $p_*(m - g \cdot m) = 0$.

(c) It suffices to find an example where $p_* : H_k(\widetilde{X}; M) \to H_k(X; M)$ is not surjective. Let $p : \mathbb{R} \to S^1$ be the universal covering $p(t) = e^{it}$. By the universal coefficient theorem,

$$H_1(S^1; M) \cong H_1(S^1; \mathbb{Z}) \otimes_{\mathbb{Z}} M \oplus \operatorname{Tor}_1^{\mathbb{Z}}(H_0(S^1), M) = \mathbb{Z} \otimes_{\mathbb{Z}} M \oplus \operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z}, M) = M,$$

since $\operatorname{Tor}_i^{\mathbb{Z}}$ vanishes when one of the entries is free (as a $\mathbb{Z}$-module) and $i > 0$. On the other hand $H_1(\mathbb{R}; M) = 0$ since $\mathbb{R}$ is contractible so $p_* = 0$ cannot be surjective.

Spring 2017-9. (a) Find the homology groups $H_k(\mathbb{RP}^2; \mathbb{Z})$ for all k.
(b) Describe a cell decomposition for $\mathbb{RP}^2 \times \mathbb{RP}^2$. Use it to show (without appealing to the Künneth theorem) that $H_3(\mathbb{RP}^2 \times \mathbb{RP}^2; \mathbb{Z})$ is nontrivial.

Hint: $H_*(\mathbb{RP}^2; \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}_{(1)} \oplus \mathbb{Z}_{(0)}$. Product CW-structure, $\ker(\partial_3) = \{(x, x) \in \mathbb{Z}^2 \mid x \in \mathbb{Z}\}$ and $\operatorname{im}(\partial_4) = \{(x, x) \in \mathbb{Z}^2 \mid x \in 2\mathbb{Z}\}$.

Referenced in: Spring 2012-10, Spring 2011-8.

(a) We give $\mathbb{RP}^2$ a CW-structure with one 0-cell, one 1-cell attached in the obvious way and then one 2-cell attached via a degree 2 map. I.e., our chain complex becomes

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0.$$

Hence, we easily compute

$$H_k(\mathbb{RP}^2; \mathbb{Z}) = \begin{cases} \mathbb{Z} & k = 0, \\ \mathbb{Z}/2\mathbb{Z} & k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Using the CW-structure described above, let $\mathbb{RP}^2 = e_0 \cup e_1 \cup e_2$ and $\mathbb{RP}^2 = f_0 \cup f_1 \cup f_2$ be the cell decomposition of the two copies of $\mathbb{RP}^2$. We know by construction that $\partial e_1 = 0$ and $\partial e_2 = 2e_1$ and similarly for f_1, f_2. Then, the product $\mathbb{RP}^2 \times \mathbb{RP}^2$ has the following CW-structure: for each $0 \leqslant i, j \leqslant 2$, an $(i+j)$-cell $e_i \times f_j$ whose boundary is $\partial(e_i \times f_j) = \partial e_i \times f_j + (-1)^i e_i \times \partial f_j$.

Hence, we have one 0-cell $e_0 \times f_0$, two 1-cells $e_1 \times f_0$ and $e_0 \times f_1$, three 2-cells $e_1 \times f_1, e_0 \times f_2$, and $e_2 \times f_0$, two 3-cells $e_1 \times f_2$ and $e_2 \times f_1$, and one 4-cell $e_2 \times f_2$. Now, we want to show that $H_3(\mathbb{RP}^2 \times \mathbb{RP}^2; \mathbb{Z}) = \frac{\ker(\partial_3)}{\operatorname{im}(\partial_4)}$ is nontrivial. For this, we use the above formulas to compute

$$\partial_3(e_1 \times f_2) = \partial e_1 \times f_2 - e_1 \times \partial f_2 = 0 \times f_2 - e_1 \times (2f_1) = -2e_1 \times f_1$$
$$\partial_3(e_2 \times f_1) = \partial e_2 \times f_1 + e_2 \times \partial f_2 = 2e_1 \times f_1 + e_2 \times 0 = 2e_1 \times f_1.$$

Hence, $\ker(\partial_3) = \{(x, x) \in \mathbb{Z}^2 \mid x \in \mathbb{Z}\}$. We also have

$$\partial_4(e_2 \times f_2) = 2e_1 \times f_2 + 2e_2 \times f_1,$$

so $\operatorname{im}(\partial_4) = \operatorname{Span}((2, 2)) = \{(x, x) \in \mathbb{Z}^2 \mid x \in 2\mathbb{Z}\}$, implying that $H_3(\mathbb{RP}^2 \times \mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z} \neq 0$. In particular, the nonzero element $[e_1 \times f_2 + e_2 \times f_1]$, which corresponds to $(1, 1) \in \ker(\partial_3)$, has boundary 0 but is not itself a boundary so is a nonzero element of $H_3(\mathbb{RP}^2 \times \mathbb{RP}^2; \mathbb{Z})$.

Spring 2017-10. Let G be a finite group and X a smooth manifold on which G acts smoothly. If the action of G on X is free (i.e., if $g \cdot x = x$ for some $x \in X$, then $g = 1$), then show that the natural quotient map

$$X \to X/G$$

is a covering map.

Hint: $\pi^{-1}(y) = \{g_1 x, \ldots, g_n x\}$. Find neighborhoods V_i of $g_i x$ so that $V_i \cong \mathbb{R}^n$ and $g_j V_i = V_k$ where $g_j g_i = g_k$.

Referenced in: Spring 2012-9.

Let $G = \{g_1, \ldots, g_n\}$, $\pi : G \to X/G$ be the natural quotient map and let $y \in X/G$. Then we can write $\pi^{-1}(y) = \{g_1 x, \ldots, g_n x\}$ for some $x \in X$ since the action is free so all of the $g_i x$ are distinct. Find charts $U_1, \ldots, U_n \subset M$ so that $g_i x \in U_i$ for all i and shrink if necessary so we may assume that they are pairwise disjoint. Define

$$V_i = \bigcap_{j=1}^{n} g_i g_j^{-1} U_j.$$

Note that we still have $g_i x \in V_i \subset U_i$ and $V_i \cap V_j = \varnothing$ if $i \neq j$. Moreover, we now have $g_j g_i^{-1} V_i = V_j$. Without loss of generality, suppose $g_1 = e$ is the identity of G so that $V_j = g_j V_1$ for all $1 \leqslant j \leqslant n$.

Now, we have $\pi(V_i) = \pi(g_i V_1) = \pi(V_1)$ for all i so define $V = \pi(V_1)$ since $\pi(gx) = \pi(x)$ for any $x \in M, g \in G$. Moreover, $\pi^{-1}(V) = \bigsqcup_{i=1}^{n} V_i$ and $V \subset X/G$ is open since projections are open maps and $V_1 \subset X$ is open. We claim that $\pi|_{V_i} : V_i \to V$ is a diffeomorphism for each i. It is surjective by definition. If $\pi(x) = \pi(y)$ for $x, y \in V_i = g_i V_1$, then there exists $g_j \in G$ so that $g_j x = y$ since G acts transitively by definition of X/G. Thus, $y \in g_j V_i = g_j g_i V_1 = V_k$ for $g_k = g_j g_i$, implying that $k = i$ since $V_k \cap V_i = \varnothing$ if $k \neq i$. Hence, $g_j g_i = g_i \implies g_j = e$ so $y = ex = x$ and $\pi|_{V_i}$ is injective. Thus, $\pi|_{V_i}$ is a diffeomorphism since it is smooth by definition of the manifold structure on X/G so π is a covering map.

Fall 2016

Fall 2016-1. Let M be a smooth manifold. Prove that for any two disjoint closed subsets $A, B \subset M$ there is a smooth function $f : M \to \mathbb{R}$ such that $f = 0$ on A and $f = 1$ on B.

Hint: Do for $\mathbb{R}^n$ using $\phi_{x_0,r}(x) = \exp((|x - x_0|^2 - r^2)^{-1})$ inside $B(x_0, r)$ and 0 outside and paracompactness. Use this to do it for $A \cap U_\alpha$ for a locally finite covering by charts and add to get $f_{A,B}$ with $f_{A,B}^{-1}(0) = A, B$ respectively.

Referenced in: Fall 2020-4.

Let $C = B(x_0, r) \subset \mathbb{R}^n$ be an open ball of radius r centered at x_0. Then, define

$$\phi_{x_0,r} : \mathbb{R}^n \to \mathbb{R}, \quad x \mapsto \begin{cases} \exp\left(\frac{1}{|x - x_0|^2 - r^2}\right) & x \in C, \\ 0 & x \notin C. \end{cases}$$

Then, $\phi_{x_0,r}$ is smooth and $\phi_{x_0,r}^{-1}(0) = \mathbb{R}^n - C$. Now, for any closed set $A \subset \mathbb{R}^n$, write $\mathbb{R}^n - A$ as a union of open balls. Since $\mathbb{R}^n$ is paracompact, this can be done in a locally finite way, so take such a union and for each ball, take its corresponding $\phi_{x,r}$ and define $\phi(y) = \sum \phi_{x,r}(y)$. By this construction, we have $\phi^{-1}(0) = A$.

Now, let $A \subset M$ be a closed set. By paracompactness of M, we can find a locally finite open cover of M by charts $\{(U_\alpha, \psi_\alpha)\}$ such that $\psi_\alpha(U_\alpha) = \mathbb{R}^n$. Then, $\psi_\alpha(A \cap U_\alpha) \subset \mathbb{R}^n$ is closed since $A \cap U_\alpha \subset U_\alpha$ is closed in the subspace topology. Hence, by the above, we may find a $\phi_\alpha : \mathbb{R}^n \to \mathbb{R}$ such that $\phi_\alpha^{-1}(0) = \psi_\alpha(A \cap U_\alpha)$.

Now, extend $\phi_\alpha \circ \psi_\alpha$ to a map $f_\alpha : M \to \mathbb{R}$ by setting it to be zero outside of U_α. Let $f_A = \sum f_\alpha$ and we have $f_A^{-1}(0) = A$. Do similarly for B and define $f : M \to \mathbb{R}$ by

$$f(x) = \frac{f_A(x)}{f_A(x) + f_B(x)}$$

and this will be our desired smooth function with $f = 0$ on A and $f = 1$ on B. This is well-defined since $A \cap B = \varnothing$ so $f_A(x) + f_B(X) \neq 0$ for all $x \in M$.

Fall 2016-2. Let $M \subset \mathbb{R}^N$ be a smooth k-dimensional submanifold. Prove that M can be immersed into $\mathbb{R}^{2k}$.

Hint: Induction. $f : M \to \mathbb{R}^L, L > 2k$, define $g : TM \to \mathbb{R}^L, g(p, v) = Df_p(v)$. This has a regular value a not in the image of g. $\pi : \mathbb{R}^L \to \mathbb{R}^{L-1}$ projection onto orthogonal complement of $\mathrm{Span}(a)$. $\pi \circ f$ is immersion.

Referenced in: Spring 2009-5.

We proceed by negative induction on N. Suppose there is an immersion $f : M \to \mathbb{R}^L$ for some $L > 0$. If $L \leqslant 2k$, then we are done because we can compose f with the standard inclusion $\mathbb{R}^L \hookrightarrow \mathbb{R}^{2k}$. So assume $L > 2k$. Define

$$g : TM \to \mathbb{R}^L, \quad (p, v) \mapsto Df_p(v).$$

By Sard's theorem, we can find a regular value a of g. Moreover, since $L > 2k$, we have $\dim(\mathbb{R}^L) > \dim(TM)$ so a cannot be in the image of g as the rank of g is always less than or equal to $2k$. Let $\pi : \mathbb{R}^L \to \mathbb{R}^{L-1}$ be the projection onto the orthogonal complement of $\mathrm{Span}(a)$. If $D(\pi \circ f)_p(v) = 0$, then

$$0 = D(\pi \circ f)_p(v) = (\pi \circ Df_p)(v) \implies Df_p(v) = g(p, v) \in \mathrm{Span}(a).$$

Since a is not in the image of g, we must have $Df_p(v) = 0$ implying that $v = 0$ since f is an immersion so Df_p is injective. Hence $D(\pi \circ f)_p$ is injective for each $p \in M$ so $\pi \circ f : M \to \mathbb{R}^{L-1}$ is an immersion, completing the inductive step.

Fall 2016-3. Let $U_1, \ldots, U_n$ be n bounded, connected, open subsets of $\mathbb{R}^n$. Prove that there exists an $(n-1)$-dimensional hyperplane $H \subset \mathbb{R}^n$ that bisects every U_i; i.e., if A and B are the two off-spaces that form $\mathbb{R}^n - H$, then

$$\mathrm{volume}(U_i \cap A) = \mathrm{volume}(U_i \cap B)$$

for all $i = 1, \ldots, n$.

Hint: Continuum of hyperplanes for each point on $S^{n-1} \subset \mathbb{R}^n$. Find one for each point that bisects U_n by intermediate value theorem. Borsuk-Ulam theorem about antipodal points being mapped to same point for $S^{n-1} \to \mathbb{R}^{n-1}$.

Let S be the unit $(n-1)$-sphere in $\mathbb{R}^n$. For each p on the surface of S, there is a continuum of oriented affine hyperplanes $\{H_p^\alpha\}_{\alpha \in \mathbb{R}}$ perpendicular to the vector from the origin to p, with the positive side of each hyperplane defined as the side pointed to by that vector. Define $g_p : \mathbb{R} \to \mathbb{R}$ by letting $g_p(\alpha)$ be the volume of U_n that is on the positive side of H_p^α. Note that this is a continuous function so by the intermediate value theorem, since there is some $\beta, \gamma \in \mathbb{R}$ so that $g_p(\beta) = \mathrm{volume}(U_n)$ and $g_p(\gamma) = 0$ (as U_n is bounded), there exists some $\alpha_p \in A$ so that $g_p(\alpha_p) = \frac{1}{2}\mathrm{volume}(U_n)$.

I.e., for each $p \in S$, we have found a hyperplane $H_p := H_p^{\alpha_p}$ that is perpendicular to the vector from the origin to p and that bisects U_n. Next, define $f : S \to \mathbb{R}^{n-1}$ component-wise by setting $f_i(p)$ to be the volume of U_i on the positive side of H_p. This function is continuous since it is continuous in each coordinate since the hyperplane chosen H_p depends continuously on p.

Hence, we may apply the Borsuk-Ulam theorem which gives us antipodal points p and q on S such that $f(p) = f(q)$. Then, the hyperplanes H_p and H_q are the same except that they are oppositely oriented. I.e., $f(p) = f(q)$ implies that the volume of U_i is the same on the positive and negative side of H_p (and H_q) for all $i = 1, \ldots, n-1$ so we are done.

Fall 2016-4. Show that

$$D = \ker(dx_3 - x_1 dx_2) \cap \ker(dx_1 - x_4 dx_2) \subset T\mathbb{R}^4$$

is a smooth distribution of rank two, and determine whether D is integrable.

Hint: Matrix with kernel D. Not integrable since $(0,0,0,1), (x_4, 1, x_1, 0) \in D$ but Lie bracket $= (1,0,0,0)$ is not.

Using the standard basis $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_4}$ of $T\mathbb{R}^4$ at each point, we note that $dx_3 - x_1 dx_2$ and $dx_1 - x_4 dx_2$ can be represented by the matrices

$$\begin{pmatrix} 0 & -x_1 & 1 & 0 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 & -x_4 & 0 & 0 \end{pmatrix},$$

respectively as maps from $T_p\mathbb{R}^4 \to \mathbb{R}$ for $p = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$. For a vector field $X = \sum_{i=1}^{f} f_i \frac{\partial}{\partial x_i}$ to be in D, i.e., in the kernel of both of these matrices, we require

$$\begin{pmatrix} 0 & -x_1 & 1 & 0 \\ 1 & -x_4 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

at all points $p \in \mathbb{R}^4$. However, we can easily see that this matrix has rank exactly two due to the 1's in column one and three so by rank-nullity, also has kernel of dimension 2 at every point. Hence, D is indeed a distribution of rank 2. Moreover, we can note that $X_1 = \frac{\partial}{\partial x_4}, X_2 = x_4 \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3} \in D$ and X_1 and X_2 are linearly independent so X_1 and X_2 form a global frame for D. However,

$$\begin{aligned} [X_1, X_2] &= \frac{\partial}{\partial x_4}\left(x_4 \frac{\partial}{\partial x_1} + 1\frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3}\right) - \left(x_4 \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3}\right)\left(1 \frac{\partial}{\partial x_4}\right) \\ &= 1\frac{\partial}{\partial x_1} + 0 + 0 - 0 - 0 - 0 = \frac{\partial}{\partial x_1} \notin D, \end{aligned}$$

so D is not integrable since it is not involutive using Frobenius' theorem.

Fall 2016-5. (a) Let M be a smooth compact manifold and $N \subset M$ a smooth compact submanifold. Explain (in terms of integrals, without reference to cohomology) what it means for a closed differential form ω to be Poincaré dual to N.
In parts (b) and (c), you are free to use your knowledge of homology and cohomology:
(b) Let $M = T^2$ be the two-dimensional torus with coordinates $(x, y) \in (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z}) \cong T^2$. Identify a submanifold $N \subset M$ Poincaré dual to the form dy, and show that they are indeed dual.
(c) Give an example of a closed 1-form on T^2 that is not Poincaré dual to any submanifold.

Hint: $\int_N i^*\eta = \int_M \eta \wedge \omega$ for all η. (b) Use $dx = \pi_1^*\theta, dy = \pi_2^*\theta$ and $N = S^1 \times \{p\}$. (c) πdx since degree is always an integer and $\int_N f^*\omega = \deg(f) \int_{S^1} \omega$ for any $f : N \to S^1$.

(a) Let $\dim(N) = n, \dim(M) = m$. The Poincaré dual to N is the unique closed $(m-n)$-form ω on M such that, for all n-forms η on M, we have

$$\int_N i^*\eta = \int_M \omega \wedge \eta,$$

where $i : N \hookrightarrow M$ is the inclusion.

(b) Denote by $\pi_1, \pi_2 : M \to S^1$ the two projections. Then, $dx = \pi_1^*\theta, dy = \pi_2^*\theta$ where θ is a 1-form on S^1 with $\int_{S^1} \theta = 1$. Note that we have $\int_M dx \wedge dy = 1$. Choose $N = S^1 \times \{p\} \subset M$ with N oriented clockwise.

Since $[dx], [dy]$ form a basis for $H^1_{dR}(M) = \mathbb{R}^2$, we may write any closed 1-form η as $[\eta] = a[dx] + b[dy]$ for some $a, b \in \mathbb{R}$. We thus check the above formula for the Poincaré dual (which is independent of representative of the cohomology class chosen) by letting $\eta = adx + bdy$. In particular, $dy \wedge \eta = -adx \wedge dy$. I.e.,

$$\int_M dy \wedge \eta = -a \int_M dx \wedge dy = -a.$$

Meanwhile, note that $i^*dx = i^*\pi_1^*\theta = (\pi_1 \circ i)^*\theta = \theta$ since $\pi_1 \circ i$ is the identity map by construction of N. We also have $i^*dy = (\pi_2 \circ i)^*\theta = 0$ since $\pi_2 \circ i$ is just the constant map $S^1 \mapsto p \in S^1$. Thus, we compute

$$\int_N i^*\eta = \int_{-S^1} a(i^*dx) + b(i^*dy) = \int_{-S^1} a\theta = -a,$$

showing that N is indeed the Poincaré dual to dy (the minus sign comes from N being oriented clockwise instead of counterclockwise).

(c) If N is any closed connected oriented 1-submanifold of M, we note that $f : N \to S^1$ satisfies $\int_N f^*\omega = \deg(f) \int_{S^1} \omega$ for ω any 1-form on S^1. In particular, $\deg(f)$ is always an integer, so we claim that the form $\alpha = \pi dx$ has no Poincaré dual. If N was the Poincaré dual of α, we would have

$$\int_N i^*(dy) = \int_M (\pi dx) \wedge dy.$$

The right-hand side is $\pi \int_M dx \wedge dy = \pi$. On the other hand,

$$\int_N i^*(dy) = \int_N i^*(\pi_2^*\theta) = \int_N (\pi_2 \circ i)^*\theta = \deg(\pi_2 \circ i) \int_{S^1} \theta = \deg(\pi_2 \circ i).$$

So $\deg(\pi_2 \circ i) = \pi$, contradicting the fact that the degree of a map is always an integer so indeed πdx has no Poincaré dual.

Fall 2016-6. Let M be a smooth, compact, oriented n-dimensional manifold. Suppose that the Euler characteristic of M is zero.

(a) Show that M admits a nowhere vanishing vector field.
(b) A *Lorentzian metric* on M is a smoothly varying, symmetric bilinear form

$$g_p : T_pM \times T_pM \to \mathbb{R}$$

of signature $(n-1, 1)$; that is, at every $p \in M$ there exists a basis $e_1, \ldots, e_n$ of T_pM such that, with respect to this basis, g_p is a diagonal matrix with $n-1$ entries of 1 and one entry of -1. Prove that M admits a Lorentzian metric.

Hint: Sum of indices of isolated zeros is zero. Put them all in a neighborhood diffeomorphic to a ball and the function on the boundary $X_p/||X_p||$ has degree zero so can be extended into the ball. Define new vector field based on this. Use nowhere vanishing vector field to define difference of two Riemannian metrics.

(a) This is exactly part (a) of Fall 2017-7.

(b) Let X be a nowhere vanishing vector field on M guaranteed by part (a). Thus, $\text{Span}(X) = \varepsilon^1$ is a trivial real line bundle on M with complementary bundle $N(X)$. In particular, we can decompose TM as $TM = N(X) \oplus \varepsilon^1$. So TM admits an indefinite metric g with signature $(n-1, 1)$ by letting $g = g_{N(X)} - g_{\varepsilon^1}$ where $g_{N(X)}$ is a Riemannian metric on $N(X)$ and g_{ε^1} is a Riemannian metric on ε^1.

Fall 2016-7. Let X be a connected CW-complex with $\pi_1(X, x)$ finite. Show that any map $X \to (S^1)^n$ is null-homotopic.

Hint: $f_*\pi_1(X,x)$ is both finite and free so must zero so f lifts to $\widetilde{f}: X \to \mathbb{R}^n$. Straight-line homotopy to constant map.

As a functor, we know that π_1 preserves products so $\pi_1((S^1)^n) = (\pi_1(S^1))^n = \mathbb{Z}^n$. In particular, $\pi_1((S^1)^n)$ is torsion-free. Now $f_*\pi_1(X,x) \subset \pi_1((S^1)^n)$ is both a subgroup of a free group and also the image of a finite group so is free and finite, implying it must be zero. Hence, letting $p: \mathbb{R}^n \to (S^1)^n$ be the universal cover, we have $f_*\pi_1(X,x) = 0 \subset p_*\pi_1(\mathbb{R}^n)$ so f lifts to a map $\widetilde{f}: X \to \mathbb{R}^n$ with $p \circ \widetilde{f} = f$.

Now, letting $f_t = t\widetilde{f}$, we see that $\widetilde{f}$ is null-homotopic in $\mathbb{R}^n$. Moreover, $p \circ f_t$ is a homotopy from a constant map $p \circ f_0 = p(0)$ to $p \circ f_1 = p \circ \widetilde{f} = f$, showing that f is null-homotopic, as desired.

Fall 2016-8. Consider $X = \mathbb{RP}^2 \vee \mathbb{RP}^2$. Let a generate π_1 of the first summand and b generate π_1 of the second summand. For $n \geqslant 1$, describe the covering space $p: Y \to X$ such that $p_*(\pi_1(Y))$ is the subgroup of $\pi_1(X)$ generated by $(ab)^n$. (A drawing and a short explanation would suffice.)

Hint: Bracelet of $2n$ S^2's which alternate between a and b.

The universal cover of $\mathbb{RP}^2$ is S^2 which is a double cover. Thus, a cover of $\mathbb{RP}^2 \vee \mathbb{RP}^2$ is a chain (finite or infinite) or bracelet of $\mathbb{RP}^2$'s and S^2's which alternatingly correspond to a and b respectively. The S^2's will be connected to two things so must occur in the middle of the chain while the $\mathbb{RP}^2$'s will attach to only one other thing so must occur at the end of the chain. In the case of $(ab)^n$, we would want $2n$ copies of S^2 in a bracelet configuration. This is because any (nontrivial) loop in this space goes along the equator of these $2n$ S^2's alternating a then b some number of times, i.e. $(ab)^n$ is the generator of its fundamental group.

Fall 2016-9. Let $S^2 \xleftarrow{q_1} S^2 \vee S^2 \xrightarrow{q_2} S^2$ be the maps that crush out one of the two summands. Let $f: S^2 \to S^2 \vee S^2$ be a map such that $q_i \circ f: S^2 \to S^2$ is a map of degree d_i. Compute the homology groups of $(S^2 \vee S^2) \cup_f D^3$.

Hint: CW structure gives chain complex with only one non-zero map. If $d_1 = d_2 = 0$, then $H_*(X) = \mathbb{Z}_{(3)} \oplus \mathbb{Z}^2_{(2)} \oplus \mathbb{Z}_{(0)}$ and otherwise, $H_*(X) = ((\mathbb{Z}/\gcd(d_1,d_2)\mathbb{Z}) \times \mathbb{Z})_{(2)} \oplus \mathbb{Z}_{(0)}$.

We note that X has the following CW-structure: one 3-cell D^3, two 2-cells e_1, e_2, and one 0-cell. So we have the chain complex

$$0 \to \mathbb{Z} \xrightarrow{\partial_3} \mathbb{Z}^2 \xrightarrow{0} 0 \xrightarrow{0} \mathbb{Z} \to 0.$$

Based on the description of the attaching maps, we have $\partial_3(D^3) = d_1e_1 + d_2e_2$ as the only nontrivial boundary map. If $d_1 = d_2 = 0$, then $\partial_3 = 0$, so we have

$$H_k(X) = \begin{cases} \mathbb{Z} & k = 0, 3, \\ \mathbb{Z}^2 & k = 2, \\ 0 & \text{otherwise.} \end{cases}$$

If d_1 and d_2 are not both zero, then ∂_3 is injective so $H_3(x) = 0$ and $H_2(X) = \operatorname{coker}(\partial_3)$. In particular, $\partial_3(a) = (d_1a, d_2a)$. By Bézout's lemma or otherwise, we can see that $\operatorname{coker}(\partial_3) \cong (\mathbb{Z}/\gcd(d_1,d_2)\mathbb{Z}) \times \mathbb{Z}$ and so we have

$$H_k(X) = \begin{cases} \mathbb{Z} & k = 0, \\ (\mathbb{Z}/\gcd(d_1,d_2)\mathbb{Z}) \times \mathbb{Z} & k = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Fall 2016-10. If $f: X \to X$ is a self-map, then the "mapping torus of f" is the quotient

$$T_f := (X \times [0,1])/(x,0) \sim (f(x),1), \forall x \in X.$$

For $n \in \mathbb{Z}$, let f_n be a degree n map on S^3. Compute the homology groups of T_{f_n}.

Hint: First, prove (or just use) general fact for $f, g : X \to Y$ to get long exact sequence $H_k(S^3) \xrightarrow{(f_n)_* - \mathrm{id}_*} H_n(S^3) \xrightarrow{j_*} H_k(T_{f_n}) \to H_{k-1}(S^3)$. Use two good pairs and naturality of homology for this. Then, answer is $H_*(T_{f_n}) = (\mathbb{Z}/(n-1)\mathbb{Z})_{(3)} \oplus \mathbb{Z}_{(1)} \oplus \mathbb{Z}_{(0)}$. If $n = 1$, get $\mathbb{Z}_{(4)}$ as well.

Referenced in: Spring 2015-9, Fall 2011-10.

We first prove a more general fact. Given $f, g : X \to Y$ continuous maps, let $Z = Y \sqcup (X \times [0,1])/\sim$ where $(x, 0) \sim f(x)$ and $(x, 1) \sim g(x)$ for all $x \in X$. Then, we claim we have a long exact sequence of the form

$$\cdots \to H_n(X) \xrightarrow{f_* - g_*} H_n(Y) \xrightarrow{j_*} H_n(Z) \to H_{n-1}(X) \to \cdots,$$

where $j : Y \to Z$ is the inclusion. Denote $I = [0,1]$ for simplicity. For this, we first consider the good pairs $(X \times I, X \times \partial I)$ and (Z, Y). Also, let $q : X \times I \to Z$ be the composition of the inclusion $X \times I \hookrightarrow Y \sqcup (X \times I)$ and the quotient $Y \sqcup (X \times I) \to Z$. By functoriality of homology, q induces a map on homology. Moreover, since q maps $X \times \partial I$ into (the image of) Y (under the quotient map) in Z, q induces a map on homology from $X \times \partial I$ to Y and thus on the relative pairs. I.e., we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{\partial_1} & H_n(X \times \partial I) & \xrightarrow{i_*} & H_n(X \times I) & \xrightarrow{\phi} & H_n(X \times I, X \times \partial I) & \xrightarrow{\partial_1} & \cdots \\ & & \downarrow & & \downarrow & & \downarrow{q_*} & & \\ \cdots & \xrightarrow{\partial_2} & H_n(Y) & \xrightarrow{j_*} & H_n(Z) & \xrightarrow{\psi} & H_n(Z, Y) & \xrightarrow{\partial_2} & \cdots \end{array}$$

Here, $i : X \times \partial I \hookrightarrow X \times I$ and $j : Y \hookrightarrow Z$ are the inclusions (j is the inclusion followed by the quotient but we can treat this as just an inclusion since the quotient doesn't identify any two things in Y). Now, we note that $q : X \times I/(X \times \partial I) \to Z/Y$ is a homeomorphism which means that $q_* : \widetilde{H}_n(X \times I/(X \times \partial I)) \to \widetilde{H}_n(Z/Y)$ is an isomorphism for all n, so since these are good pairs, where relative homology is equal to the reduced homology of the quotient, we have that

$$q_* : H_n(X \times I, X \times \partial I) \to H_n(Z, Y)$$

is an isomorphism for all n. Next, observe that $X \times I$ deformation retracts to either $X \times \{0\}$ or $X \times \{1\}$. So the inclusions $X \times \{0\} \hookrightarrow X \times I$ and $X \times \{1\} \hookrightarrow X \times I$ induce isomorphisms

$$H_n(X \times \partial I) \cong H_n(X \times \{0\}) \oplus H_n(X \times \{1\}) = H_n(X) \oplus H_n(X)$$

and the map

$$i_* : H_n(X \times \partial I) = H_n(X) \oplus H_n(X) \to H_n(X \times I) \cong H_n(X)$$

is surjective, with $i_*(a, b) = a + b$. By exactness of the top row, we thus get $\phi = 0$ and ∂_1 is injective. I.e., $H_n(X \times I, X \times \partial I)$ is isomorphic to its image in $H_{n-1}(X \times \partial I)$ via ∂_1 and this image is precisely $\ker(i_*)$. But we know

$$\ker(i_*) = \{(a, -a) \in H_{n-1}(X) \oplus H_{n-1}(X) \mid a \in H_{n-1}(X)\} \cong H_{n-1}(X).$$

Hence, we have $H_n(Z, Y) \cong H_n(X \times I, X \times \partial I) = \ker(i_*) \cong H_{n-1}(X)$ so the bottom row is our desired exact sequence and it suffices to show that $\partial_2 = f_* - g_*$. To see this, note that $\partial_2 : H_n(Z, Y) \cong H_{n-1}(X) \to H_{n-1}(Y)$ is the composition $q_* \circ \partial_1$ by commutativity which is just

$$q_* : H_{n-1}(X \times \partial I) = H_{n-1}(X) \oplus H_{n-1}(X) \to H_{n-1}(Y)$$

restricted to the image of ∂_1 which is $\ker(i_*)$. This map is the sum of the two maps $H_n(X \times \{0\}) \to H_n(Y)$ and $H_n(X \times \{1\}) \to H_n(Y)$ induced by f and g respectively. I.e., the map $\ker(i_*) \to H_n(Y)$ is $(a, -a) \mapsto f_*(a) + g_*(-a) = (f_* - g_*)(a)$ so we get the desired long exact sequence.

Returning to the question at hand, we apply the above long exact sequence with $X = Y = S^3, f = f_n, g = \mathrm{id}$ so that $Z = T_{f_n}$ is what we are trying to compute the homology of:

$$\cdots \to H_k(S^3) \xrightarrow{(f_n)_* - \mathrm{id}_*} H_k(S^3) \xrightarrow{j_*} H_k(T_{f_n}) \to H_{k-1}(S^3) \to \cdots.$$

Using the fact that $H_k(S^3) = \mathbb{Z}$ when $k = 0$ or 3 and 0 otherwise, this simplifies to

$$\cdots \to 0 \xrightarrow{j_*} H_4(T_{f_n}) \to \mathbb{Z} \xrightarrow{(f_n)_* - \mathrm{id}_*} \mathbb{Z} \xrightarrow{j_*} H_3(T_{f_n}) \to 0 \to \cdots$$
$$\to 0 \to H_1(T_{f_n}) \to \mathbb{Z} \xrightarrow{(f_n)_* - \mathrm{id}_*} \mathbb{Z} \xrightarrow{j_*} H_0(T_{f_n}) \to 0.$$

First, T_{f_n} is clearly connected so $H_0(T_{f_n}) \cong \mathbb{Z}$, implying that on zeroth homology, j_* is surjective so an isomorphism and $(f_n)_* - \mathrm{id}_*$ is 0 and thus $H_1(T_{f_n}) \cong \mathbb{Z}$. The left part of this sequence tells us that on third homology, $H_4(T_{f_n}) \cong \ker((f_n)_* - \mathrm{id}_*)$ and $H_3(T_{f_n}) \cong \operatorname{coker}((f_n)_* - \mathrm{id}_*)$.

However, $(f_n)_* - \mathrm{id}_* : H_3(S^3) \to H_3(S^3)$ is precisely multiplication by the degree of f_n minus the degree of id which is $n - 1$. Thus, $H_4(T_{f_n}) = 0$ and $H_3(T_{f_n}) = \mathbb{Z}/(n-1)\mathbb{Z}$. To summarize, we have

$$H_k(T_{f_n}) = \begin{cases} \mathbb{Z} & k = 0, 1, \text{ or } k = 4 \text{ and } n = 1, \\ \mathbb{Z}/(n-1)\mathbb{Z} & k = 3, \\ 0 & \text{otherwise.} \end{cases}$$

Spring 2016

Spring 2016-1. Consider the space of all straight lines in $\mathbb{R}^2$ (not necessarily those passing through the origin). Explain how to give it the structure of a smooth manifold. Is it orientable?

Hint: $f : X \to \mathbb{RP}^2$, $\quad ax + by + c = 0 \mapsto [a : b : c] \in \mathbb{RP}^2$ is a bijection onto its image which is $\mathbb{RP}^2 - \{[0:0:1]\}$. Not orientable since $\mathbb{RP}^2$ is not, double cover $S^2 \to \mathbb{RP}^2$.

Referenced in: Fall 2023-1.

Let X be the space of all straight lines in $\mathbb{R}^2$. We have a map

$$f : X \to \mathbb{RP}^2, \quad ax + by + c = 0 \mapsto [a : b : c] \in \mathbb{RP}^2.$$

f is well-defined since the numbers $a, b, c \in \mathbb{R}$ are determined by a line in $\mathbb{R}^2$ up to scaling (exactly as in $\mathbb{RP}^2$). Further, f is injective since each equation $ax + by + c = 0$ determines a unique straight line in $\mathbb{R}^2$. Finally, we can see that the image of f is $U = \mathbb{RP}^2 - \{[0:0:1]\}$ since $c = 0$ is not a line for any $c \in \mathbb{R}$. We know that $U \subset \mathbb{RP}^2$ is an open subset.

Thus, $X \cong U$ is an open submanifold of $\mathbb{RP}^2$, so we give it the smooth manifold structure inherited from being an open subset of $\mathbb{RP}^2$. Now, we know that $\mathbb{RP}^2$ is non-orientable and has orientation double cover $p : S^2 \to \mathbb{RP}^2$. Then $Y = S^2 - \{(0,0,1), (0,0,-1)\}$, which is still orientable as it is an open subset of S^2, is the orientation double cover of U via $p|_Y : Y \to U$. Note that $Y \cong \mathbb{R}^2 - \{(0,0,0)\}$ is still connected so U must be non-orientable.

Spring 2016-2. Let X and Y be submanifolds of $\mathbb{R}^n$. Prove that, for almost every $a \in \mathbb{R}^n$, the translate $X + a$ intersects Y transversely.

Hint: Show $F : X \times \mathbb{R}^n \to \mathbb{R}^n, F(x, a) = x + a$ is transverse to Y. Thom's transversality theorem.

Referenced in: Fall 2023-4, Fall 2010-2.

Define $F : X \times \mathbb{R}^n \to \mathbb{R}^n$ by $F(x, a) = x + a$. We claim F is transverse to Y. To show this, by definition, it suffices to show that for each $(x, a) \in F^{-1}(Y)$,

$$dF_{(x,a)}(T_{(x,a)}(X \times \mathbb{R}^n)) + T_{F(x,a)}Y = T_{F(x,a)}\mathbb{R}^n = \mathbb{R}^n.$$

In fact, we claim that just the first term already gives us all of $\mathbb{R}^n$. Note that

$$dF_{(x,a)} : T_{(x,a)}(X \times \mathbb{R}^n) \to T_{F(x,a)}\mathbb{R}^n = \mathbb{R}^n$$

is a linear map between vector spaces. For $b \in \mathbb{R}^n$ arbitrary, consider the curve $\gamma(t) = (x, a+bt)$ which passes through (x, a) at time $t = 0$ and has $\gamma'(0) = (0, b)$. So $(0, b) \in T_{(x,a)}(X \times \mathbb{R}^n) = T_xX \times T_a\mathbb{R}^n = T_xX \times \mathbb{R}^n$. Then

$$dF_{(x,a)}(0, b) = \frac{d}{dt}|_{t=0}(F \circ \gamma)(t) = \frac{d}{dt}(x + a + bt) = b \in \mathbb{R}^n,$$

so, since b was arbitrary, $dF_{(x,a)}$ is surjective for any $x \in X, a \in \mathbb{R}^n$, showing the claim. Now, by Thom's transversality theorem, for almost all $a \in \mathbb{R}^n$, the map $F_a : X \to \mathbb{R}^n$ given by $F_a(x) = F(x, a)$ is transverse to Y. But $F_a(X) = X + a$ so we conclude that $X + a$ is transverse to Y for almost all $a \in \mathbb{R}^n$.

Spring 2016-3. Consider the vector field $X(z) = z^{2016} + 2016z^{2015} + 2016$ on $\mathbb{C} = \mathbb{R}^2$. (By this we mean the following: take a complex coordinate z on $\mathbb{C}$, identify $T_z\mathbb{C} = \mathbb{C}$, and write $X(z) = z^{2016} + 2016z^{2015} + 2016 \in T_z\mathbb{C}$.) Compute the sum of the indices of X over all zeros of X.

Hint: Poincaré-Hope does **not** apply. Sum multiplicities of roots. 2016.

The polynomial $z^{2016} + 2016z^{2015} + 2016 = 0$ has only finitely many zeros (at most 2016 distinct zeros) so all of its roots are isolated. Since $\mathbb{C}$ is algebraically closed, we can factor $X(z)$ as

$$X(z) = \prod_{i=1}^{k}(z - z_i)^{m_i},$$

where $z_i \in \mathbb{C}$ are the zeroes of X and $m_i \in \mathbb{N}$ are the corresponding multiplicities. Moreover, the local index $\text{ind}_{z_i}(X) = m_i$ is the multiplicity of the root since we can locally write X as $X(z) = (z - z_i)^{m_i} \cdot h(z)$ for $h(z)$ some non-vanishing polynomial. Since the multiplicities of the roots of a complex polynomial sum to its degree (by the fundamental theorem of algebra), the sum of the indices of X over all zeros of X is 2016.

Spring 2016-4. Let M be a compact odd-dimensional manifold with nonempty boundary ∂M. Show that the Euler characteristics of M and ∂M are related by:

$$\chi(M) = \frac{1}{2}\chi(\partial M).$$

Hint: Glue two copies of M together along the boundary to make N. N is closed, odd-dimensional so has $\chi(N) = 0$ by Poincaré duality (use $\mathbb{Z}/2\mathbb{Z}$ coefficients) and then use Mayer-Vietoris to get equation.

This is exactly Spring 2022-9.

Spring 2016-5. Let M be a compact oriented manifold of dimension n with de Rham cohomology group $H^1_{dR}(M; \mathbb{R}) = 0$ and let T^n be the n-dimensional torus. For which integers k does there exist a smooth map $f : M \to T^n$ of degree k?

Hint: Must have degree 0. Grading 1 cohomology classes generate all the other ones as exterior algebra, i.e., $H^n_{dR}(T^n)$ is generated by $\theta_1 \wedge \cdots \wedge \theta_n$.

Let $f : M \to T^n$. This induces $f^* : H^*_{dR}(T^n) \to H^*_{dR}(M)$. We know that if $\theta_1, \ldots, \theta_n$ are the cohomology classes that generate $H^1_{dR}(T^n)$, then $H^n_{dR}(T^n)$ is generated by $\theta_1 \wedge \cdots \wedge \theta_n$. Then, we have

$$f^*(\theta_1 \wedge \cdots \wedge \theta_n) = f^*(\theta_1) \wedge \cdots \wedge f^*(\theta_n).$$

Then if $H^1_{dR}(M) = 0$, we have $f^*(\theta_i) = 0$ for all $1 \leqslant i \leqslant n$, implying that $f^*(\theta_1 \wedge \cdots \wedge \theta_n) = 0$. Hence $f^* : H^n_{dR}(T^n) \to H^n_{dR}(M)$ is the zero homomorphism so f has degree 0.

Spring 2016-6. Let $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the two-dimensional torus with coordinates $(x,y) \in \mathbb{R}^2$ and let $p \in T^2$.

(a) Compute the de Rham cohomology of the punctured torus $T^2 - \{p\}$.
(b) Is the volume form $\omega = dx \wedge dy$ exact on $T^2 - \{p\}$?

Hint: $T^2 - \{p\}$ deformation retracts onto $S^1 \vee S^1$. $H^*_{dR}(T^2 - \{p\}) = \mathbb{R}^2_{(1)} \oplus \mathbb{R}_{(0)}$. (b) is yes since top form and second cohomology is zero.

(a) It is not hard to see that $T^2 - \{p\}$ deformation retracts onto $S^1 \vee S^1$ so $H^*_{dR}(T^2 - \{p\}) \cong H^*_{dR}(S^1 \vee S^1)$ and hence we have (via de Rham's theorem and the universal coefficient theorem)

$$H^k_{dR}(T^2 - \{p\}) = \begin{cases} \mathbb{R} & k = 0, \\ \mathbb{R}^2 & k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Yes. ω is a top form (since $T^2 - \{p\}$ is dimension 2) so is closed. Then, by part (a), $H^2_{dR}(T^2 - \{p\}) = 0$ implies that all closed 2-forms are exact so ω is exact.

Spring 2016-7. Exhibit a space whose fundamental group is isomorphic to $(\mathbb{Z}/m\mathbb{Z}) * (\mathbb{Z}/n\mathbb{Z})$, where $\mathbb{Z}/k\mathbb{Z}$ denotes the integers modulo k and $*$ denotes the free product. Exhibit another space whose fundamental group is isomorphic to $(\mathbb{Z}/m\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z})$.

Hint: Attach via $z \mapsto z^k$ to get $\mathbb{Z}/k\mathbb{Z}$. Wedge sum and product of these spaces.

Referenced in: Spring 2024-7.

For $k \in \mathbb{N}$, construct a space X_k by attaching a 2-cell as follows: regarding $D^2 \subset \mathbb{C}$ as the unit disk in the complex plane, and S^1 as the unit circle, attach D^2 to S^1 along $\partial D^2 = S^1$ via the map $z \mapsto z^k$. Then, it is clear that $\pi_1(X_k) = \langle a \mid a^k \rangle = \mathbb{Z}/k\mathbb{Z}$. By Van Kampen's, $\pi_1(X_m \vee X_n) = (\mathbb{Z}/m\mathbb{Z}) * (\mathbb{Z}/n\mathbb{Z})$ and since π_1 is a functor that preserves products, $\pi_1(X_m \times X_n) = (\mathbb{Z}/m\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z})$.

Spring 2016-8. Let L_x be the x-axis, L_y be the y-axis, and L_z be the z-axis of $\mathbb{R}^3$. Compute

$$\pi_1(\mathbb{R}^3 - L_x - L_y - L_z).$$

Hint: $\mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$. Deformation retract to S^2 minus 6 points and then homotopy to $\mathbb{R}^2$ minus 5 points and finally to the wedge sum of 5 copies of S^1. Use induction/Van Kampen's to prove.

This is exactly Fall 2022-7 with $n = 3$.

Spring 2016-9. Let X be a topological space and $p \in X$. The *reduced suspension* ΣX of X is the space obtained from $X \times [0,1]$ by contracting $(X \times \{0,1\}) \cup (\{p\} \times [0,1])$ to a point. Describe the relation between the homology groups of X and ΣX.

Hint: $\widetilde{H}_n(\Sigma X) = \widetilde{H}_n(SX) = \widetilde{H}_{n-1}(X)$ for all $n \geqslant 1$ and $H_0(\Sigma X) = \mathbb{Z}$. Because $\Sigma X = SX/(\{p\} \times [0,1])$ so look at long exact sequence. Get $\widetilde{H}_n(SX)$ from Mayer-Vietoris.

Referenced in: Spring 2024-6.

Let $SX = X \times [0,1]/\sim$ where $(x,0) \sim (y,0)$ and $(x,1) \sim (y,1)$ for all $x, y \in X$ be the (unreduced) suspension of X. Then, $\Sigma X = SX/(\{p\} \times [0,1])$ and $(SX, \{p\} \times [0,1])$ is a good pair so we have a long exact sequence

$$\cdots \to \widetilde{H}_n(\{p\} \times [0,1]) \to \widetilde{H}_n(SX) \to \widetilde{H}_n(\Sigma X) \to \widetilde{H}_{n-1}(\{p\} \times [0,1]) \to \cdots,$$

since $H_n(SX, \{p\} \times [0,1]) = \widetilde{H}_n(SX/(\{p\} \times [0,1])) = \widetilde{H}_n(\Sigma X)$ for all n. But then, $\{p\} \times [0,1] \cong [0,1]$ is contractible so $\widetilde{H}_n(\{p\} \times [0,1]) = 0$ for all n, implying that $\widetilde{H}_n(SX) = \widetilde{H}_n(\Sigma X)$ for all $n \geqslant 1$. Then, by Fall 2020-6, we know $\widetilde{H}_n(SX) = \widetilde{H}_{n-1}(X)$ for all $n \geqslant 1$ and for $n = 0, H_n(\Sigma X) = \mathbb{Z}$ since ΣX is connected.

Spring 2016-10. Consider the 3-form on $\mathbb{R}^4$ given by

$$\alpha = x_1 dx_2 \wedge dx_3 \wedge dx_4 - x_2 dx_1 \wedge dx_3 \wedge dx_4 + x_3 dx_1 \wedge dx_2 \wedge dx_4 - x_4 dx_1 \wedge dx_2 \wedge dx_3.$$

Let S^3 be the unit sphere in $\mathbb{R}^4$ and let $\iota : S^3 \to \mathbb{R}^4$ be the inclusion map.

(a) Evaluate $\int_{S^3} \iota^*\alpha$.

(b) Let γ be the 3-form on $\mathbb{R}^4 - \{0\}$ given by:

$$\gamma = \frac{\alpha}{(x_1^2 + x_2^2 + x_3^2 + x_4^2)^k}$$

for $k \in \mathbb{R}$. Determine the values of k for which γ is closed and those for which it is exact.

Hint: Stokes', $4\text{vol}(B) = 2\pi^2$. Never exact, closed for $k = 2$. Use $R = x_1^2 + x_2^2 + x_3^2 + x_4^2$ and differentiate $d\gamma$.

(a) Note that $d\alpha = 4dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 = 4dV$ where dV is the standard volume form on $\mathbb{R}^4$. Next, we recall that $\partial B^4 = S^3$ where B^4 is the closed unit ball in $\mathbb{R}^4$. Thus, by Stokes' theorem,

$$\int_{S^3} \iota^*\alpha = \int_B d\alpha = \int_B 4dV = 4\text{vol}(B).$$

We know the volume of a unit n-ball can be calculated recursively as $V(B^n) = \frac{2\pi}{n} V(B^{n-2})$ and the unit 2-ball or closed unit disk has volume π so $\int_{S^3} \iota^*\alpha = 4 \cdot \frac{2\pi}{4}\pi = 2\pi^2$.

(b) Let $R = x_1^2 + x_2^2 + x_3^2 + x_4^2$ so that $\gamma = \alpha R^{-k}$. Noting that $dR = \sum_{i=1}^4 2x_i dx_i$, we have

$$\begin{aligned} d\gamma &= d(R^{-k}) \wedge \alpha + R^{-k} d\alpha \\ &= -kR^{-k-1} \sum_{i=1}^4 2x_i dx_i \wedge \alpha + 4R^{-k} dV \\ &= -2kR^{-k-1} \sum_{i=1}^4 x_i^2 dV + 4R^{-k} dV \\ &= (4 - 2k)R^{-k} dV. \end{aligned}$$

Since dV is nowhere vanishing and $R \neq 0, d\gamma = 0$ if and only if $4 = 2k$, i.e., $k = 2$. For exactness, note that for any $p \in S^3$, we have $\gamma_p = \alpha_p$ so $\iota^*\gamma = \iota^*\alpha$ and thus

$$\int_{S^3} \iota^*\gamma = \int_{S^3} \iota^*\alpha = 2\pi^2.$$

But if γ is exact, say with $d\theta = \gamma$, then $d(\iota^*\theta) = \iota^* d\theta = \iota^*\gamma$ so

$$2\pi^2 = \int_{S^3} \iota^*\gamma = \int_{S^3} d(\iota^*\theta) = \int_{\partial S^3} \iota^*\theta = 0,$$

by Stokes' theorem, since $\partial S^3 = \varnothing$, a contradiction. So no value of k makes γ exact.

Fall 2015

Fall 2015-1. Let $M_n(\mathbb{R})$ be the space of $n \times n$ matrices with real coefficients.

(a) Show that $SL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det(A) = 1\}$ is a smooth submanifold of $M_n(\mathbb{R})$.

(b) Show that $SL(n, \mathbb{R})$ has trivial Euler characteristic.

Hint: Preimage theorem with $A \mapsto \det(A)$. Homotopy equivalent to $SO(n)$ which is parallelizable so has nowhere vanishing vector field. Also, compact so has $\chi = 0$ by Poincaré-Hopf.

Referenced in: Fall 2010-3.

(a) Consider $\det : M_n(\mathbb{R}) \to \mathbb{R}$. Note that $SL(n, \mathbb{R}) = \det^{-1}(1)$ so we claim that $1 \in \mathbb{R}$ is a regular value of $\det$. Let $A \in \det^{-1}(1)$. Let $c \in \mathbb{R}$. For $B \in T_A M_n(\mathbb{R}) \cong M_n(\mathbb{R})$, we have

$$\begin{aligned} d(\det)_A(B) &= \lim_{h\to 0} \frac{\det(A + hB) - \det(A)}{h} = \lim_{h\to 0} \frac{\det(A)(\det(I + hA^{-1}B) - 1)}{h} \\ &= \lim_{h\to 0} \frac{\det(I + hA^{-1}B)}{h}, \end{aligned}$$

so if we take $B = \frac{c}{n}A$, then

$$d(\det)_A(\frac{c}{n}A) = \lim_{h\to 0} \frac{\det(I + h\frac{c}{n}I) - 1}{h} = \lim_{h\to 0} \frac{(1 + h\frac{c}{n})^n - 1}{h} = \lim_{h\to 0} \frac{c}{n}n(1 + h\frac{c}{n})^{n-1} = \frac{c}{n}n = c,$$

so $d(\det)_A$ is indeed surjective. So $SL_n(\mathbb{R})$ is a smooth manifold by the preimage theorem.

(b) This is exactly Fall 2020-10.

Fall 2015-2. Let $f, g : M \to N$ be smooth maps between smooth manifolds that are smoothly homotopic. Prove that if ω is a closed form on N, then $f^*\omega$ and $g^*\omega$ are cohomologous.

Hint: Prove $\mathrm{id} - \pi^* i^* = d\Phi + \Phi d$ where $\Phi : \Omega^k(M \times \mathbb{R}) \to \Omega^{k-1}(M \times \mathbb{R})$ is $\Phi(\sum_I a_I(x,t)dx^I + \sum_J b_J(x,t)dt \wedge dx^J) = \sum_J \left(\int_0^t b_J(x,s)ds\right) dx^J$. So $i_0^* = i_1^*$ on the level of cohomology for $i_j : M \to M \times \mathbb{R}$ is given by $i_j(p) = (p, j)$.

Referenced in: Spring 2020-1.

We first prove the following lemma. Let M be a smooth manifold, $\pi : M \times \mathbb{R} \to M$ the projection onto the first factor, and $i : M \to M \times \mathbb{R}$ the inclusion $p \mapsto (p, 0)$. Consider the induced maps

$$\pi^* : H^*_{dR}(M) \to H^*_{dR}(M \times \mathbb{R}), \quad i^* : H^*_{dR}(M \times \mathbb{R}) \to H^*_{dR}(M).$$

Then there exists a cochain homotopy

$$\Phi : \Omega^k(M \times \mathbb{R}) \to \Omega^{k-1}(M \times \mathbb{R})$$

between id and $\pi^* i^*$, i.e.,

$$\mathrm{id} - \pi^* i^* = d\Phi + \Phi d \text{ on } \Omega^k(M \times \mathbb{R}).$$

In particular, π^* is an isomorphism with inverse i^*.

Given an arbitrary k-form $\omega \in \Omega^k(M \times \mathbb{R})$, take local coordinates and write

$$\omega = \sum_I a_I(x,t)dx^I + \sum_J b_J(x,t)dt \wedge dx^J$$

where the first sum is over increasing multi-indices I of length k and the second is over increasing multi-indices J of length $k - 1$. Define Φ by

$$\Phi(\omega) = \sum_J \left(\int_0^t b_J(x,s)ds\right) dx^J.$$

We claim that

$$d(\Phi(\omega)) + \Phi(d\omega) = \omega - \pi^* i^* \omega.$$

By linearity, we may separate into the following cases: $\omega = a(x,t)dx^I$, and $\omega = b(x,t)dt \wedge dx^J$. In the former case, $\Phi(\omega) = 0$ and

$$\Phi(d\omega) = \left(\int_0^t \frac{\partial a}{\partial s} ds\right) dx^I = (a(x,t) - a(x,0))dx^I = \omega - \pi^* i^* \omega.$$

In the latter case, $i^*\omega = 0$, so $(\mathrm{id} - \pi^* i^*)\omega = \omega$, and we compute

$$\begin{aligned} d(\Phi(\omega)) &= d\left(\int_0^t b(x,s)ds\right) dx^J = \left(\int_0^t \frac{\partial b}{\partial s} ds\right) dx^J + \left(\int_0^t \sum_{m=1}^n \frac{\partial b}{\partial x^m} dx^m ds\right) \wedge dx^J \\ &= \omega + \sum_{m=1}^n \left(\int_0^t \frac{\partial b}{\partial x^m} ds\right) dx^m \wedge dx^J \quad \text{by the F.T.C.,} \\ \Phi(d\omega) &= \Phi\left(-\sum_{m=1}^n \frac{\partial b}{\partial x^m} dt \wedge dx^m \wedge dx^J\right) \\ &= -\sum_{m=1}^n \left(\int_0^t \frac{\partial b}{\partial x^m} ds\right) dx^m \wedge dx^J. \end{aligned}$$

Thus, $d(\Phi(\omega)) + \Phi(d\omega) = \omega$, showing the claim in local coordinates. But the definition of Φ is independent of the choice of coordinates, since it only affects the real coordinate. We have thus shown that id and $i \circ \pi$ induce chain homotopic maps on the de Rham complex of $M \times \mathbb{R}$ so π^* is an isomorphism on cohomology with i^* as its inverse.

Back to our problem, let $F : M \times \mathbb{R} \to N$ be the homotopy between f and g so that $F(x,0) = f(x), F(x,1) = g(x)$. Define $i_0, i_1 : M \to M \times \mathbb{R}$ by $i_0(p) = (p,0)$ and $i_1(p) = (p,1)$. Clearly, we have $f = F \circ i_0$ and $g = F \circ i_1$. By the above lemma, we have shown that $i_0^* = i_1^* = (\pi^*)^{-1}$ (by replacing 0 with 1) so $f^* = g^*$ as maps on cohomology so $f^*\omega$ and $g^*\omega$ are cohomologous for ω a closed form.

Fall 2015-3. For two smooth vector fields X, Y on a smooth manifold M, prove the formula

$$[L_X, i_Y]\omega = i_{[X,Y]}\omega,$$

where L_X is the Lie derivative in the direction of X, i_X is the interior product of X, and ω is a k-form for $k \geqslant 1$.

Hint: Compute left hand side using $(\iota_Y \omega)(V_1, \ldots, V_{k-1}) = \omega(Y, V_1, \ldots, V_{k-1})$ and $(L_X\omega)(V_1, \ldots, V_k) = X(\omega(V_1, \ldots, V_k)) - \sum_{i=1}^k \omega(V_1, \ldots, V_{i-1}, [X, V_i], V_{i+1}, \ldots, V_k)$.

This is exactly Spring 2020-4.

Fall 2015-4. Let $M = \mathbb{R}^3/\mathbb{Z}^3$ be a three dimensional torus and $C = \pi(L)$, where $L \subset \mathbb{R}^3$ is the oriented line segment from $(0,1,1)$ to $(1,3,5)$ and $\pi : \mathbb{R}^3 \to M$ is the quotient map. Find a differential form on M which represents the Poincaré dual of C.

Hint: $\eta = 4xd \wedge dy - 2dx \wedge dz + dy \wedge dz$. Expand and solve in terms of the generators of the exterior algebra dx, dy, and dz.

Referenced in: Spring 2014-5.

Identify L with $\pi(L)$ for simplicity. We seek a 2-form η such that, for any 1-form θ, we have

$$\int_L \theta = \int_M \eta \wedge \theta.$$

Now, L is a loop on M that wraps in the x-direction once, in the y-direction twice, and in the z-direction four times. We know that the coordinate 1-forms on T^3, dx, dy, dz, generate the de Rham cohomology of T^3 as an exterior algebra. Hence, we can write $\theta = adx + bdy + cdz$ for some $a, b, c \in \mathbb{R}$ and $\eta = Adx \wedge dy + Bdx \wedge dz + Cdy \wedge dz$ for some $A, B, C \in \mathbb{R}$. Calculating the integrals, we have

$$\int_L adx + bdy + cdz = a + 2b + 4c,$$

$$\begin{aligned}\int_M (Adx \wedge dy + Bdx \wedge dz + Cdy \wedge dz) \wedge (adx + bdy + cdz) &= \int_M (Ac - Bb + Ca)dx \wedge dy \wedge dz \\ &= Ac - Bb + Ca.\end{aligned}$$

This immediately gives $A = 4, C = 1$, and $B = -2$. Hence the Poincaré dual is

$$\eta = 4dx \wedge dy - 2dx \wedge dz + dy \wedge dz.$$

Fall 2015-5. Recall that the Hopf fibration $\pi : S^3 \to S^2$ is defined as follows: if we identify

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$$

and $S^2 = \mathbb{CP}^1$ with homogeneous coordinates $[z_1, z_2]$, then $\pi(z_1, z_2) = [z_1, z_2]$. Show that π does not admit a section, i.e., a smooth map $s : S^2 \to S^3$ such that $\pi \circ s = \mathrm{id}_{S^2}$.

Hint: Just look at induced maps on homology, $H_2(S^2) = \mathbb{Z}$ while $H_2(S^3) = 0$.

If $s : S^2 \to S^3$ with $\pi \circ s = \mathrm{id}_{S^2}$ existed, this would induce a map $s_* : H_2(S^2) \to H_2(S^3)$ on homology such that $\pi_* \circ s_* = (\pi \circ s)_* = (\mathrm{id}_{S^2})_* = \mathrm{id}_{H_2(S^2)}$. Note that $H_2(S^2) = \mathbb{Z}$ while $H_2(S^3) = 0$. Thus $s_* = 0$ but $\mathrm{id}_{\mathbb{Z}} \neq 0$, a contradiction.

Fall 2015-6. Let $M^m \subset \mathbb{R}^n$ be a smooth submanifold of dimension $m < n-2$. Show that its complement $\mathbb{R}^n - M$ is connected and simply-connected.

Hint: Homotope a path in $\mathbb{R}^n$ to one that is transversal to M and show this must not intersect M. Use extension theorem on a homotopy from $[0,1] \times [0,1] \to M$ with $C = \{0,1\} \times [0,1] \cup [0,1] \times \{0,1\}$ a closed subset with the image of $H|_C$ transverse to M.

Referenced in: Spring 2024-3, Fall 2012-3.

We showed that $M - \mathbb{R}^n$ is (path-)connected in part (a) of Spring 2023-5. For simply-connectedness, let $\gamma : [0,1] \to \mathbb{R}^n - M$ be a loop with $\gamma(0) = \gamma(1) = p$. Since γ is nullhomotopic in $\mathbb{R}^n$ ($\mathbb{R}^n$ is simply connected), let $H : [0,1] \times [0,1]$ be a path homotopy between γ and the constant map at p. Namely, $H(0,x) = \gamma(x), H(1,x) = p$, and $H(t,0) = H(t,1) = p$ for all $t, x \in [0,1]$.

Let $C = \{0,1\} \times [0,1] \cup [0,1] \times \{0,1\}$, which is a closed subset of $[0,1] \times [0,1]$. Note that the image of $H|_C$ does not intersect $M \subset \mathbb{R}^n$ so $H|_C$ is trivially transverse to M. So by the extension theorem, we can find a map $G : [0,1] \times [0,1] \to \mathbb{R}^n$ such that $G = H$ on a neighborhood of C and G is also transverse to M. I.e., for any $(t,x) \in G^{-1}(M)$, we have

$$dG_{(t,x)}(T_{(t,x)}([0,1] \times [0,1])) + T_{G(t,x)}M = T_{G(t,x)}\mathbb{R}^n.$$

But the left-hand side has dimension at most

$$\dim(T_{(t,x)}([0,1] \times [0,1])) + \dim(T_{G(t,x)}M) = \dim([0,1] \times [0,1]) + \dim(M) = 2 + m < 2 + n - 2 = n,$$

while the right hand side clearly has dimension n. Hence, $G^{-1}(M)$ must in fact be empty so G maps to $\mathbb{R}^n - M$. Namely, G is a path homotopy between γ and a constant map in $\mathbb{R}^n - M$ so γ is nullhomotopic. But this was true for arbitrary γ so $\pi_1(\mathbb{R}^n - M) = 0$ and $\mathbb{R}^n - M$ is simply connected.

Fall 2015-7. Show that there exists no smooth degree one map from $S^2 \times S^2$ to $\mathbb{CP}^2$.

Hint: Cohomology rings $H^*(\mathbb{CP}^2) = \mathbb{Z}[\alpha]/(\alpha^3), |\alpha| = 2$ and $H^*(S^2 \times S^2) = \mathbb{Z}[\beta, \gamma]/(\beta^2, \gamma^2), |\beta| = |\gamma| = 2$. Write $f^*(\alpha) = a\beta + b\gamma$ so $f^*(\alpha^2) = 2ab\beta\gamma$ so degree is $2ab$.

This is exactly Spring 2020-6.

Fall 2015-8. Show that $\mathbb{CP}^{2n}, n \in \mathbb{Z}^+$, is not a covering space of any manifold except itself.

Hint: $\pi_1(X)$ acts on $\mathbb{CP}^{2n}$ via deck transformations. Any map $g : \mathbb{CP}^{2n} \to \mathbb{CP}^{2n}$ has a fixed point by Lefschetz trace formula so they are all the identity.

This is exactly Spring 2020-5.

Fall 2015-9. Given a continuous map $f : X \to Y$ between topological spaces, define

$$C_f = \Big((X \times [0,1]) \bigsqcup Y\Big) / \sim,$$

where $(x, 1) \sim f(x)$ for all $x \in X$ and $(x, 0) \sim (x', 0)$ for all $x, x' \in X$. Here $\bigsqcup$ is the disjoint union. Show that there is a long exact sequence

$$\cdots \to H_{i+1}(X) \xrightarrow{f_*} H_{i+1}(Y) \to \widetilde{H}_{i+1}(C_f) \to H_i(X) \xrightarrow{f_*} H_i(Y) \to \cdots,$$

where f_* is the map on homology induced from f and $\widetilde{H}_i$ denotes the ith reduced homology group.

Hint: Good pair long exact sequence with mapping cylinder and mapping cone. Consider what happens if we replace $A \hookrightarrow M_f$ by $X \hookrightarrow A \hookrightarrow M_f$, i.e., changing $H_i(A)$ to $H_i(X), H_i(M_f)$ to $H_i(Y)$ and i_* to f_*.

This is exactly the first part of Fall 2022-9.

Fall 2015-10. Let $\mathbb{RP}^n$ be the real projective space given by $S^n/\sim$, where $S^n = \{||x|| = 1\} \subset \mathbb{R}^{n+1}$ and $x \sim -x$ for all $x \in S^n$.

(a) Give a cell (CW) decomposition of $\mathbb{RP}^n$ for $n \geqslant 1$.
(b) Use the cell decomposition to compute the homology groups $H_k(\mathbb{RP}^n), k \geqslant 0$.
(c) For which values of $n \geqslant 1$ is $\mathbb{RP}^n$ orientable? Explain.

Hint: One cell in each dimension with double cover for attaching maps. $H_k(\mathbb{RP}^n) = \mathbb{Z}$ if $k = n$ and n is odd, $\mathbb{Z}/2\mathbb{Z}$ if $k \leqslant n$ and k is odd and 0 otherwise. Orientable if and only if top homology is $\mathbb{Z}$ if and only if n is odd.

(a) and (b) This is done in Fall 2020-7 and we get

$$H_k(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & k = 0, k = n \text{ and } n \text{ odd}, \\ \mathbb{Z}/2\mathbb{Z} & 1 \leqslant k < n, k \text{ odd}, \\ 0 & \text{otherwise.} \end{cases}$$

(c) We know that a closed and connected n-manifold M is orientable if and only if $H_n(M) \cong \mathbb{Z}$. In our cases, $\mathbb{RP}^n$ is closed and connected so is orientable if and only if n is odd.

Spring 2015

Spring 2015-1. Let $M(n, m, k) \subset M(n, m)$ denote the space of $n \times m$-matrices of rank k. Show that $M(n, m, k)$ is a smooth manifold of dimension $nm - (n - k)(m - k)$.

Hint: $\begin{pmatrix} I_k & 0 \\ -CA^{-1} & I_{n-k} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix}$ so consider $f : N \to M(n - k, m - k)$ given by $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto D - CA^{-1}B$. Preimage theorem.

This is exactly Fall 2018-2 with $n \times m$ instead of $n \times n$ (but this only changes the dimensions throughout).

Spring 2015-2. Assume that $N \subset M$ is a codimension 1 properly embedded submanifold. Show that N can be written as $f^{-1}(0)$, where 0 is a regular value of a smooth function $f : M \to \mathbb{R}$, if and only if there is a vector field X on M that is transverse to N.

Hint: Riemannian metric g, then $df_p(v) = g((\nabla f)_p, v)$ and ∇f is a vector field transverse to N. Converse is actually **false**: $S^1 \subset S^1 \times S^1$ would have f always positive or negative on $M - N$.

Suppose $N = f^{-1}(0)$ where 0 is a regular value of $f : M \to \mathbb{R}$. Let g be a Riemannian metric on M. Define the gradient vector field ∇f on M as the dual vector field to the 1-form df with respect to the metric g. I.e., $df_p(v) = g((\nabla f)_p, v)$ for all $v \in T_pM$. Since f is constant on N, we have $df \equiv 0$ on $TN \subset TM$ so $\nabla f \perp TN$.

We also note that $(\nabla f)_p \neq 0$ for all $p \in N$ since 0 is a regular value of f so df cannot vanish at p. Now, since N has codimension 1, it follows that $T_pN + \text{Span}(\nabla f)_p = T_pM$ for all $p \in N$ by considering dimension. Hence, ∇f is our desired transverse vector field.

In fact, the converse is false. Let $M = S^1 \times S^1$ with coordinates (θ, ϕ), $N \subset M$ the circle with coordinates $(\theta, 0)$, and $X = \frac{\partial}{\partial \phi}$. So X is transverse to N. Also, $M - N$ is connected so if $f : M \to \mathbb{R}$ has $N = f^{-1}(0)$, then f is either always positive or negative on $M - N$. In particular, N is the set of maxima or minima of f, which means the output of f at points in N cannot be regular values since the partial derivatives are zero.

Spring 2015-3. Consider two collections of 1-forms $\omega_1, \ldots, \omega_k$ and $\phi_1, \ldots, \phi_k$ on an n-dimensional manifold M. Assume that

$$\omega_1 \wedge \cdots \wedge \omega_k = \phi_1 \wedge \cdots \wedge \phi_k$$

never vanishes on M. Show that there are smooth functions $f_{ij} : M \to \mathbb{R}$ such that

$$\omega_i = \sum_{j=1}^{k} f_{ij}\phi_j, \quad i = 1, \ldots, k.$$

Hint: Wedge product of 1-forms is 0 if and only if the 1-forms are linearly dependent.

By Spring 2014-4, we know that a wedge of 1-forms is 0 if and only if the 1-forms are linearly dependent. Thus, $\phi_1 \wedge \cdots \wedge \phi_k \neq 0$ so $\phi_1, \ldots, \phi_k$ are linearly independent while $\phi_1 \wedge \cdots \phi_k \wedge \omega_i = \omega_1 \wedge \cdots \wedge \omega_k \wedge \omega_i = 0$ for any $1 \leqslant i \leqslant k$ since $\omega_i \wedge \omega_i = 0$ for 1-forms. Thus, we see that ω_i must be a linear combination of $\phi_1, \ldots, \phi_k$ as desired.

Spring 2015-4. Consider a smooth map $F : \mathbb{RP}^n \to \mathbb{RP}^n$.

(a) When n is even show that F has a fixed point.
(b) When n is odd give an example where F does not have a fixed point.

Hint: $H^k(\mathbb{RP}^n;\mathbb{Q})$ is $\mathbb{Q}$ in degree 0 only so use Lefschetz trace formula. Rotation of $S^1 = \mathbb{RP}^1$.

(a) For n even, we know the homology of $\mathbb{RP}^n$ is

$$H_k(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & k = 0, \\ \mathbb{Z}/2\mathbb{Z} & 1 \leqslant k < n, k \text{ odd}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, by the universal coefficients theorem, since the Ext terms vanish (as $\mathbb{Q}$ is an injective $\mathbb{Z}$-module) and $\mathbb{Q}$ is torsion-free, we have

$$H^k(\mathbb{RP}^n;\mathbb{Q}) = \begin{cases} \mathbb{Q} & k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Using the Lefschetz trace formula, we know

$$L(f) = \sum_{i=0}^{n} (-1)^i \mathrm{Tr}(F^* : H^i(\mathbb{RP}^n;\mathbb{Q}) \to H^i(\mathbb{RP}^n;\mathbb{Q})) = 1,$$

since only the $i = 0$ term is non-zero and F certainly induces an isomorphism in degree 0 as F^* is a cohomology ring homomorphism. Thus, by the Lefschetz fixed point theorem, F has a fixed point.

(b) We can just take $n = 1, \mathbb{RP}^1 = S^1$ and F rotation by $\pi/2$ has no fixed points.

Spring 2015-5. Assume we have a codimension 1 distribution $\Delta \subset TM$.

(a) Show if the quotient bundle TM/Δ is trivial (or equivalently that there is a vector field on M that never lies in Δ), then there is a 1-form ω on M such that $\Delta = \ker\omega$ everywhere on M.
(b) Give an example where TM/Δ is not trivial.
(c) With ω_1 and ω_2 as in (a) show that $\omega_1 \wedge d\omega_1 = f^2\omega_2 \wedge d\omega_2$ for a smooth function f that never vanishes.
(d) If ω is as in (a) and $\omega \wedge d\omega \neq 0$, show that Δ is not integrable.

Hint: $q : TM \to TM/\Delta, \phi : TM/\Delta \to M \times \mathbb{R}$ Define $\omega_p = \pi \circ \phi_p \circ q_p$. (b) Möbius strip with $\mathrm{Span}(\frac{\partial}{\partial x})$. (c) Show $\omega_1 = f\omega_2$ since $I(\Delta)$ is generated by ω_2 (d) $\ker(\omega) = \mathrm{span}(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^{m-1}}), \omega|_U = f_U dx^m$. Set $\alpha|_U = \frac{df_U}{f_U}$ and put together with partition of unity.

(a) Let $q : TM \to TM/\Delta$ be the fiber-wise surjection, $\phi : TM/\Delta \to M \times \mathbb{R}$ be the isomorphism that comes from triviality of TM/Δ, and $\pi : M \times \mathbb{R} \to \mathbb{R}$ be projection onto the second factor. For each $p \in M$, define

$$\omega_p(X) = \pi(\phi_p(q_p(X)), \quad \text{i.e., } \omega_p = \pi \circ \phi_p \circ q_p.$$

Then, $\omega_p(X) = 0$ if and only if $\phi_p(q_p(X)) \in M \times \{0\}$. But we know $\phi_p : T_pM/\Delta_p \to \{p\} \times \mathbb{R}$ maps into $\{p\} \times \mathbb{R}$ so $\omega_p(X) = 0$ if and only if $\phi_p(q_p(X)) = (p, 0)$ which occurs if and only if $q_p(X) = 0$ in T_pM/Δ_p, i.e., if and only if $X_p \in \Delta_p$. Thus, $\Delta = \ker(\omega)$.

(b) Take $M = [0, 2\pi] \times [-1, 1]/\sim$ to be the Möbius strip and $\Delta = \mathrm{Span}(\frac{\partial}{\partial x})$. Suppose that $\omega \in \Omega^1(M)$ satisfies $\Delta = \ker(\omega)$, then ω is of the form $\omega = f(x, y)dy$. Moreover, we know that $f(0, y) = -f(2\pi, -y)$ since ω is invariant under $(0, y) \sim (2\pi, -y)$. Restricting to the segment $[0, 2\pi] \times \{0\}$, we see that $f(0, 0) = -f(2\pi, 0)$ which by the intermediate value theorem, implies that f has a zero on $[0, 2\pi] \times \{0\}$. At this point, $\ker(\omega)$ has dimension 2, contradicting $\ker(\omega) = \Delta$. Thus, by part (a), this is an example where TM/Δ cannot be trivial.

(c) It suffices to show that $\omega_1 = f\omega_2$ where f does not vanish, since then, $d\omega_1 = df \wedge \omega_2 + fd\omega_2$ so $\omega_1 \wedge d\omega_1 = f\omega_2 \wedge (df \wedge \omega_2 + fd\omega_2) = f^2\omega_2 \wedge d\omega_2$. To show this, we note that the ideal $I(\Delta)$, consisting of forms that vanish on Δ, is generated by ω_1 so we can write $\omega_2 = f\omega_1$. If $f(p) = 0$ for some p, then $(\omega_2)_p = f(p)(\omega_1)_p = 0$, contradicting the fact that $\ker(\omega_2)$ is always codimension 1 so indeed this f is nonvanishing.

(d) This is exactly (a) $\implies$ (b) of Fall 2022-4.

Spring 2015-6. Let
$$\omega = \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}$$
be a 2-form defined on $\mathbb{R}^3 - \{0\}$. If $i : S^2 = \{x^2 + y^2 + z^2 = 1\} \to \mathbb{R}^3$ is the inclusion, then compute $\int_{S^2} i^*\omega$. Also compute $\int_{S^2} j^*\omega$, where $j : S^2 \to \mathbb{R}^3$ maps $(x, y, z) \mapsto (3x, 2y, 8z)$.

Hint: Both are 4π. Use Stokes. Show that ω is closed so $\int_E k^*(d\omega) = 0$ where $E = j(S^2) - S^2$ and $k : E \hookrightarrow \mathbb{R}^3 - \{0\}$.

This is exactly Spring 2023-3 (the different numbers in part (b) do not change the solution).

Spring 2015- 7. Define the de Rham cohomology groups $H^i_{dR}(M)$ of a manifold M and compute $H^i_{dR}(S^1)$, $S^1 = \mathbb{R}/\mathbb{Z}, i = 0, 1, \ldots,$ directly from the definition.

Hint: $H^*_{dR}(S^1) = \mathbb{R}_{(0)} \oplus \mathbb{R}_{(1)}$. Exterior derivative cochain complex. Smooth 1-periodic functions and $fd\theta$ for θ the angular coordinate.

Referenced in: Fall 2011-6.

Let $\Omega^k(M)$ be the k-forms on M. We have the de Rham cochain complex
$$0 \to \Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} \Omega^2(M) \xrightarrow{d_2} \cdots,$$
where d_i is the exterior derivative and is defined locally by
$$d_k(fdx_{i_1} \wedge \cdots \wedge dx_{i_k}) = df \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}, \text{ where } df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i.$$
The de Rham cohomology groups $H^k_{dR}(M)$ are the cohomology groups of this cochain complex, i.e.,
$$H^k_{dR}(M) = \frac{\ker(d_k)}{\operatorname{im}(d_{k-1})}.$$
Since S^1 is 1-dimensional, there are no k-forms for $k > 1$ so we need only determine $H^0_{dR}(S^1)$ and $H^1_{dR}(S^1)$ as all other cohomology groups will be zero. I.e., in this case, the de Rham cochain complex is simply
$$0 \to \Omega^0(S^1) \xrightarrow{d} \Omega^1(S^1) \to 0.$$
Now, $\Omega^0(S^1) = C^\infty(S^1)$ consists of the smooth functions on S^1, which are precisely 1-periodic functions $\mathbb{R} \to \mathbb{R}$. Next, $\Omega^1(S^1)$ consists of 1-forms $fd\theta$, where f is a 1-periodic smooth function and $d\theta$ is the usual angular coordinate 1-form.

For $f \in \Omega^0(S^1)$, we have $df = \frac{\partial f}{\partial \theta} d\theta$ so $H^0_{dR}(S^1) = \ker(d) \subset \Omega^0(S^1)$ consists of those 1-periodic functions that are constant in θ, and hence is isomorphic to the vector space of real constants $\mathbb{R}$. On the other hand $\operatorname{im}(d) \subset \Omega^1(S^1)$ consists of forms $fd\theta$ where f is a derivative of a function in $C^\infty(S^1)$, i.e., those functions whose integral is 1-periodic.

So we see that $\operatorname{im}(d) = \{fd\theta \mid f \in C^\infty(S^1), \int_0^1 f(\theta)d\theta = 0\}$. Thus, $fd\theta, gd\theta \in H^1_{dR}(S^1) = \frac{\Omega^1(S^1)}{\operatorname{im}(d)}$ are equal on the level of cohomology if and only $\int_0^1 f(\theta) - g(\theta)d\theta = 0$. So each cohomology class is uniquely represented by its integral which means the first cohomology is also isomorphic to $\mathbb{R}$. To summarize,
$$H^k_{dR}(S^1) = \begin{cases} \mathbb{R} & k = 0, 1, \\ 0 & \text{otherwise.} \end{cases}$$

Spring 2015-8. Let X be a CW complex consisting of one vertex p, two edges a and b, and two 2-cells f_1 and f_2 where the boundaries of a and b map to p, the boundary of f_1 is mapped to the loop ab^2 (that is first a and then b twice), and the boundary of f_2 is mapped to the loop ba^2.

(a) Compute the fundamental group $\pi_1(X)$ of X. Is it a finite group?
(b) Compute the homology groups $H_i(X), i = 0, 1, \ldots,$ of X.

Hint: Group presentation. $\pi_1(X) = \mathbb{Z}/3\mathbb{Z}$. $H_*(X;\mathbb{Z}) = \mathbb{Z}_{(0)} \oplus \mathbb{Z}/3\mathbb{Z}_{(1)}$.

In essence, this is exactly Spring 2022-7 but with the cubes replaced by squares. In this case $\pi_1(X) = \mathbb{Z}/3\mathbb{Z}$ and

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0, \\ \mathbb{Z}/3\mathbb{Z} & n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Spring 2015-9. Let X, Y be topological spaces and let $f, g : X \to Y$ be two continuous maps. Consider the space Z obtained from the disjoint union $(X \times [0,1]) \sqcup Y$ by identifying $(x, 0) \sim f(x)$ and $(x, 1) \sim g(x)$ for all $x \in X$. Show that there is a long exact sequence of the form:

$$\cdots \to H_i(X) \xrightarrow{a} H_i(Y) \xrightarrow{b} H_i(Z) \xrightarrow{c} H_{i-1}(X) \to \cdots.$$

Also describe the maps a, b, c.

Hint: Two long exact sequences for the pairs $(X \times I, X \times \partial I)$ and (Z, Y) where the relatively homology fits in with an isomorphism. $a = f_* - g_*, b = j_*$ for $j : Y \hookrightarrow Z$, and $c = \partial_1$ is the first coordinate of the connecting map induced from the snake lemma.

This was proved in Fall 2016-10 except we did not describe c. But $c : H_n(Z) \to H_{n-1}(X)$ arises from the connecting map $\partial : H_n(Z) \to H_{n-1}(X) \oplus H_{n-1}(X)$ obtained via the snake lemma.

Spring 2015-10. Let $n \geqslant 0$ be an integer. Let M be a compact, orientable, smooth manifold of dimension $4n + 2$. Show that $\dim(H^{2n+1}(M;\mathbb{R}))$ is even.

Hint: $H^{2n+1}(M) \times H^{2n+1}(M) \to \mathbb{R}$ by $(\omega, \eta) \mapsto \int_M \omega \wedge \eta$. Show this is bilinear, anti-symmetric, and non-degenerate. Corresponds to an invertible $k \times k$ ($k = \dim(H^{2n+1}(M))$) matrix with $A = -A^T$ but if k is odd, we get $\det(A) = -\det(A)$, a contradiction.

This is exactly Fall 2021-5.

Fall 2014

Fall 2014-1. Let $f : M \to N$ be a proper immersion between connected manifolds of the same dimension. Show that f is a covering map.

Hint: Submersion is open map so onto. Local diffeomorphism implies locally even covering and then piece together using compact/connected.

Since f is an immersion between manifolds of the same dimension, it is also a submersion. Moreover, f is proper so it is a closed map and thus $f(M) \subset N$ is closed. The rest of the argument is essentially the same as Spring 2018-1.

Fall 2014-2. Let $M^m \subset \mathbb{R}^n$ be a closed connected submanifold of dimension m.

(a) Show that $\mathbb{R}^n - M^m$ is connected when $m \leqslant n-2$.
(b) When $m = n-1$, show that $\mathbb{R}^n - M^m$ is disconnected by showing that the mod 2 intersection number $I_2(f, M) = 0$ for all smooth maps $f : S^1 \to \mathbb{R}^n$.

Hint: Homotope a path in $\mathbb{R}^n$ to one that is transversal to M and show this must not intersect M. Slice chart with one 0. Take linear path between $(x_1, \ldots, x_m, \varepsilon)$ and $(x_1, \ldots, x_m, -\varepsilon)$ and make into loop to get contradiction.

This is exactly Spring 2023-5.

Fall 2014-3. Let ω be an n-form on a closed connected non-orientable n-manifold M and $\pi : \mathcal{O} \to M$ the orientation cover.

(a) Show that $\pi^*\omega$ is exact.
(b) Show that ω is exact.

Hint: Show $\pi^*\omega$ integrates to 0 since it equals its negative as $\deg(F) = -1$. (b) π^* is injective: $U \subset Y$ evenly covered by $f^{-1}(U) = \bigcup_{i=1}^k U_i$ with local inverses $\phi_i : U \to U_i$ to f. Then, define $g(\omega)|_U = \frac{1}{k}\sum_{i=1}^k \phi_i^*\omega$ which is well-defined and satisfies $g \circ f^* = \text{id}$.

(a) Since M is connected, closed, and non-orientable, $\mathcal{O}$ is connected, closed, and orientable. Thus its top de Rham cohomology is $H^n_{dR}(\mathcal{O}) \cong \mathbb{R}$ with isomorphism

$$[\theta] \leftrightarrow \int_{\mathcal{O}} \theta \in \mathbb{R},$$

by de Rham's theorem. In particular, note that a closed n-form is exact (i.e., zero in cohomology) if and only if θ integrates to 0 over $\mathcal{O}$. So now, we claim that $\int_{\mathcal{O}} \pi^*\omega = 0$. As $\pi : \mathcal{O} \to M$ is the orientation double cover, the group of deck transformations of π is $\{\text{id}, F\}$, where F is the unique non-identity covering transformation and is orientation-reversing. So $\deg(F) = -1$ and

$$\int_{\mathcal{O}} F^*(\pi^*\omega) = \deg(F) \int_{\mathcal{O}} \pi^*\omega = -\int_{\mathcal{O}} \pi^*\omega.$$

On the other hand, because F is a deck transformation, $\pi \circ F = \pi$ so $F^*\pi^* = (\pi \circ F)^* = \pi^*$. Hence,

$$\int_{\mathcal{O}} \pi^*\omega = \int_{\mathcal{O}} F^*\pi^*\omega = -\int_{\mathcal{O}} \pi^*\omega,$$

so we must have $\int_{\mathcal{O}} \pi^*\omega = 0$, showing that $\pi^*\omega$ is indeed exact.

(b) Since π is a finite covering map, π^* is injective on top cohomology by Spring 2019-6. By part (a), $\pi^*\omega = 0 \in H^n_{dR}(\mathcal{O})$ so $\omega = 0 \in H^n_{dR}(M)$ so ω is exact.

Fall 2014-4. Show that for $n \geqslant 1$ any smooth map $f : S^{n-1} \to S^{n-1}$ has a smooth extension $F : D^n \to D^n$ where $D^n = \{x \in \mathbb{R}^n \mid |x| \leqslant 1\}$.

Hint: Bump function.

Let $\phi : [0, \infty) \to [0, 1]$ be a smooth bump function supported in $[\frac{1}{2}, \frac{3}{2}]$ which is 1 on $[\frac{3}{4}, \frac{5}{4}]$. Define $\Phi : \mathbb{R}^n \to [0, 1]$ as the radially symmetric function with $\Phi(x) = \phi(|x|)$ so that Φ is supported in the annulus $\{\frac{1}{2} \leqslant |x| \leqslant \frac{3}{2}\}$ and is 1 on the annulus $\{\frac{3}{4} \leqslant |x| \leqslant \frac{5}{4}\}$. Then,

$$F : D^n \to D^n, \quad F(x) = \begin{cases} \Phi(x) f(\frac{x}{||x||}) & 0 < ||x|| \leqslant 1, \\ 0 & x = 0, \end{cases}$$

is the desired smooth extension of f.

Fall 2014-5. Let M be a smooth manifold and ω a nowhere vanishing 1-form on M. Show that ω is locally proportional to the differential of a function (i.e., around every point $p \in M$ there is a neighborhood $U \ni p$ and functions $f, \lambda : U \to \mathbb{R}$ such that $\omega = \lambda df$) if and only if $\omega \wedge d\omega = 0$.

Hint: Work locally. $\ker(\omega)$ is an integrable distribution which is the zero locus of a function f. Show ω_q and df_q have same kernel for any q in a neighborhood.

If $\omega = \lambda df$ locally near p, then $d\omega = d\lambda \wedge df$ can be computed locally. So $\omega \wedge d\omega = \lambda df \wedge (d\lambda \wedge df) = 0$ since $df \wedge df = 0$ for df a 1-form. Conversely, suppose $\omega \wedge d\omega = 0$. So by Fall 2022-4, we know $\ker(\omega)$ is an integrable codimension 1 distribution.

So at every point $p \in M$, there is an immersed integral submanifold N through p, i.e., $T_pN = \ker(\omega_p)$. An immersed submanifold of codimension 1 can locally be expressed as the zero locus of a function, i.e., there is a neighborhood $U \ni p$ and a function f on U so that $N \cap U = \{q \in U \mid f(q) = 0\}$. Then $df_q(X) = 0$ for any $X \in T_qN = \ker(\omega_q)$ so $\ker(\omega_q) = \ker(df_q)$, implying that $\lambda(q)df_q = \omega_q$ for some $\lambda(q) \neq 0$, showing $\omega = \lambda df$.

Fall 2014-6. Recall that the *rank* of a matrix is the dimension of the span of its row vectors. Show that the space of all 2×3 matrices of rank 1 forms a smooth manifold.

Hint: $\begin{pmatrix} I_1 & 0 \\ -CA^{-1} & I_1 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix}$ so consider $f : N \to M_{1\times 2}$ given by $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto D - CA^{-1}B$. Preimage theorem.

This is a slightly more general case of Fall 2018-2 where we have $n \times m$ instead of $n \times n$ but this doesn't change anything nontrivial.

Fall 2014-7. A compact surface of genus g, smoothly embedded in $\mathbb{R}^3$, bounds a compact region called a *handlebody* H.

(a) Prove that two copies of H glued together along their boundaries by the identity map produces a closed topological 3-manifold M.
(b) Compute the homology of M.
(c) Compute the relative homology of (M, H), where H is one of the two copies.

Hint: $H_*(M) = \mathbb{Z}_{(3)} \oplus \mathbb{Z}^g_{(2)} \oplus \mathbb{Z}^g_{(1)} \oplus \mathbb{Z}_{(0)}, H_*(M,H) = \mathbb{Z}_{(3)} \oplus \mathbb{Z}^g_{(2)}$. Good pair (X, H) where we know $H_*(X,H) = H_*(H, \partial H)$ and we compute $H_*(H, \partial H)$ using Lefschetz duality/universal coefficient formula from $H_k(H)$. The important map $\mathbb{Z}^g \to \mathbb{Z}^g$ is zero. Or can Mayer-Vietoris instead.

(a) If $x \notin \partial H$, then $x \in H$ is contained in some chart $x \in U \subset H$ with $U \cong \mathbb{R}^3$ since H is a 3-manifold. We may also assume that $U \cap \partial H = 0$ by shrinking as necessary. Then $(x, 0) \in (U, 0)$ and $(x, 1) \in (U, 1)$ are both contained in charts with each (U, i) open and homeomorphic to U and thus to $\mathbb{R}^3$. On the other hand, if $x \in \partial H$, then pick some $V \subset H$ open with $V \cong H^3$, the upper plane of $\mathbb{R}^3$. Then, $(x, 0) = (x, 1) \in M$ has neighborhood $(V, 0) \cup (V, 1)$ which is homeomorphic to two upper half planes glued together at the boundary which is exactly $\mathbb{R}^3$ itself. Thus, every point in M has a neighborhood homeomorphic to $\mathbb{R}^3$ so M is a 3-manifold without boundary. It is compact since it is the quotient of a compact space $H \times \{0, 1\}$ as H is compact.

(b) We did this in Fall 2017-9 with three copies of H and the idea is the exact same. The only difference is that $H_2(M, H) = \mathbb{Z}^g$ and $H_3(M, H) = \mathbb{Z}$ so the homology groups become

$$H_k(M) = \begin{cases} \mathbb{Z} & k = 0, 3, \\ \mathbb{Z}^g & k = 1, 2, \\ 0 & \text{otherwise.} \end{cases}$$

(c) In fact, we computed this in the process of doing part (b). We have

$$H_k(M,H) = \begin{cases} \mathbb{Z}^g & k=2, \\ \mathbb{Z} & k=3, \\ 0 & \text{otherwise.} \end{cases}$$

Fall 2014-8. Consider the space $X = M_1 \cup M_2$, where M_1 and M_2 are Möbius bands and $M_1 \cap M_2 = \partial M_1 = \partial M_2$. Here a *Möbius band* is the quotient space $([-1,1]\times[-1,1])/((1,y)\sim(-1,-y))$.

(a) Determine the fundamental group of X.
(b) Is X homotopy equivalent to a compact orientable surface of genus g for some g?

Hint: Klein bottle. $\pi_1(X) = \langle a, b \mid bab^{-1}a\rangle \cong \mathbb{Z} \rtimes \mathbb{Z}$. Polygon representation, CW structure.

(a) This is exactly Fall 2017-8.

(b) No. $H_1(X) = \pi_1(X)^{ab} \cong \mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ while $H_1(M_g)\cong\mathbb{Z}^{2g}$ where M_g is the compact orientable surface of genus g. Since X and M_g have different homology groups, they can never be homotopy equivalent.

Fall 2014-9. Determine all the connected covering spaces of the wedge sum $\mathbb{RP}^{14}\vee\mathbb{RP}^{15}$.

Hint: Alternate between S^{14} and S^{15}. Finite chain with $\mathbb{RP}$ on both sides. Finite bracelet of spheres. Infinite chain in one or two directions.

First, note that $\mathbb{RP}^n$ has universal double cover S^n for any n. In particular, no space other than $\mathbb{RP}^n$ and S^n can cover $\mathbb{RP}^n$ as this would correspond to a nontrivial, proper subgroup of $\mathbb{Z}/2\mathbb{Z}$ which doesn't exist. Now, in a covering space of $\mathbb{RP}^{14}\vee\mathbb{RP}^{15}$, when we have an S^{14} or S^{15}, there are two connecting points that can be wedge summed with coverings of the other while when we have an $\mathbb{RP}^{14}$ or $\mathbb{RP}^{15}$, there is one connecting point.

Thus, we have the following options. First, a finite chain that begins with $\mathbb{RP}^{14}$ or $\mathbb{RP}^{15}$, ends with the other one, and has alternating S^{14}'s and S^{15}'s in between. Second, a finite alternating bracelet of S^{14}'s and S^{15}'s. Third, a (one-sided infinite) chain that starts with $\mathbb{RP}^{14}$ or $\mathbb{RP}^{15}$ and has alternating S^{14}'s and S^{15}'s infinitely in one direction. Fourth, a (two-sided infinite) chain of alternating S^{14}'s and S^{15}'s going in both directions.

Fall 2014-10. Let D be the unit disk in the complex plane and S^1 be the unit circle in the complex plane. Consider the 2-dimensional torus $T^2 = S^1\times S^1$ and two copies D_1 and D_2 of D. Let X be the quotient of the disjoint union $T^2 \sqcup D_1 \sqcup D_2$ by the equivalence relations

$$e^{i\theta}\sim(e^{ip\theta},1), \quad e^{i\phi}\sim(1,e^{iq\phi}),$$

where $e^{i\theta}\in D_1, e^{i\phi}\in D_2, (e^{i\theta},1),(1,e^{iq\phi})\in T^2$, and p,q are integers >1. Compute the homology groups of X.

Hint: $H_*(X) = \mathbb{Z}_{(2)}\oplus(\mathbb{Z}/p\mathbb{Z}\times\mathbb{Z}/q\mathbb{Z})_{(1)}\oplus\mathbb{Z}_{(0)}$. CW structure with two 1-cells and three 2-cells attached via $aba^{-1}b^{-1}, a^p, b^q$ respectively.

Note that X is the space with CW structure as follows: one 0-cell v, two 1-cells a and b, and three 2-cells f_1, f_2, and f_3. The 1-cells are attached to v in the obvious way and f_1 is attached via the loop $aba^{-1}b^{-1}$ and f_2, f_3 are attached via the loops a^p, b^q respectively. Hence, the cell complex is

$$0\to\mathbb{Z}^3\xrightarrow{\partial_2}\mathbb{Z}^2\xrightarrow{\partial_1}\mathbb{Z}\to 0.$$

Based on the attaching maps, we know $\partial_1 = 0$ while $\partial_2(f_1) = a + b - a - b = 0, \partial_2(f_2) = p \cdot a, \partial_2(f_3) = q \cdot b$. In other words $\partial_2(x, y, z) = (py, qz)$. So $H_2(X) = \ker(\partial_2) \cong \mathbb{Z}, H_1(X) = \mathbb{Z}^2/\text{im}(\partial_2) \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$, and clearly $H_0(X) = \mathbb{Z}$. To summarize,

$$H_k(X) = \begin{cases} \mathbb{Z} & k = 0, 2, \\ \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z} & k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Spring 2014

Spring 2014-1. Let $\Gamma \subset \mathbb{R}^2$ be the graph of the function $y = |x|$.

(a) Construct a smooth function $f : \mathbb{R} \to \mathbb{R}^2$ whose image is Γ.
(b) Can f be an immersion?

Hint: Bump function, $f(x) = (\psi(x)x, \psi(x)|x|)$. Compute $|f'(x_0)|$ from both directions.

(a) Let $\phi : \mathbb{R} \to [0, 1]$ be a bump function supported in $[-2, 2]$ and identically 1 on $K = [-1, 1]$. Let $\psi = 1 - \phi$ so that $\psi : \mathbb{R} \to [0, 1]$ is 0 on K and 1 outside $[-2, 2]$. Define

$$f : \mathbb{R} \to \mathbb{R}^2, \quad x \mapsto (\psi(x)x, \psi(x)|x|).$$

Now, f is smooth at $x \neq 0$ since ψ and $|\cdot|$ both are. f is also smooth at $x = 0$ since $\psi(x)x \equiv 0$ in a neighborhood of 0. Note that $\psi(x)|x| = |\psi(x)x|$ since $\psi(x) \geqslant 0$ for all x. Thus, the image of f is $\{(y, |y|) \mid y = \psi(x)x, x \in \mathbb{R}\}$ so $\text{im}(f) \subset \Gamma$. But it is easy to see by construction (and the intermediate value theorem) that $x \mapsto \psi(x)x$ is a surjective function so $\text{im}(f) = \Gamma$ as desired.

(b) It is clear that any such map f, must be of the form $f(x) = (g(x), |g(x)|)$ for some smooth function $g : \mathbb{R} \to \mathbb{R}$. Since $(0, 0) \in \Gamma$, there exists $x_0 \in \mathbb{R}$ such that $g(x_0) = 0$. If f is an immersion, then df_{x_0} is injective so $df_{x_0} \neq 0$ and $\frac{d}{dx}|_{x=x_0}|g(x)|$ must exist for smoothness. But we can compute

$$\begin{aligned} g'(x_0) &= \lim_{h\to 0} \frac{g(x_0+h) - g(x_0)}{h} = \lim_{h\to 0^\pm} \frac{g(x_0+h)}{h}, \\ |g'(x_0)| &= \lim_{h\to 0^+} \frac{|g(x_0+h)|}{h} = \lim_{h\to 0^+} \frac{|g(x_0+h)| - |g(x_0)|}{h} = \frac{d}{dx}\bigg|_{x=x_0} |g(x)|, \\ |g'(x_0)| &= \lim_{h\to 0^-} \frac{|g(x_0+h)|}{|h|} = \lim_{h\to 0^-} \frac{-|g(x_0+h)| + |g(x_0)|}{h} = -\frac{d}{dx}\bigg|_{x=x_0} |g(x)|. \end{aligned}$$

So we conclude that $g'(x_0) = \frac{d}{dx}\big|_{x=x_0}|g(x)| = 0$, a contradiction and f cannot be an immersion.

Spring 2014-2. Let W be a smooth manifold with boundary, and $f : \partial W \to \mathbb{R}^n$ a smooth map, for some $n \geqslant 1$. Show that there exists a smooth map $F : W \to \mathbb{R}^n$ such that $F|_{\partial W} = f$.

Hint: Neighborhood $\partial W \subset U \subset W$ diffeomorphic to $[0, 1) \times \partial W$. Projection $\pi : U \to \partial W$. Bump function ϕ, $F(x) = \phi(x)f(\pi(x))$.

Choose a (collar) neighborhood U of ∂W in W that is diffeomorphic to $[0, 1) \times \partial W$ (with ∂W corresponding to $\{0\} \times \partial W$). Write $\pi : U \to \partial W$ as the projection of U onto ∂W. Let $\phi : W \to [0, 1]$ be a bump function supported in U and identically 1 on ∂W, which can be done since $\partial W \subset W$ is closed. Define $F : W \to \mathbb{R}^n$ by $F(x) = \phi(x)f(\pi(x))$. This is well-defined since ϕ is zero outside of U and $\pi(x)$ is defined for $x \in U$. This agrees with f on the boundary ∂W.

Spring 2014-3. Let $S^n \subset \mathbb{R}^{n+1}$ be the unit sphere. Determine the values of $n \geqslant 0$ for which the antipodal map $S^n \to S^n$, $x \mapsto -x$ is isotopic to the identity.

Hint: Degree is homotopy invariant, $\deg(A) = (-1)^{n+1}$. Block diagonal matrix with blocks rotation by πt for $0 \leqslant t \leqslant 1$.

This is exactly Spring 2019-3.

Spring 2014-4. Let $\omega_1, \cdots, \omega_k$ be 1-forms on a smooth n-dimensional manifold M. Show that $\{\omega_i\}$ are linearly independent if and only if

$$\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_k \neq 0.$$

Hint: Linear combination and expand. Dual vector fields are linearly independent.

Referenced in: Spring 2021-10, Spring 2015-3.

Suppose that $\omega_1, \ldots, \omega_k$ are linearly dependent. Without loss of generality, suppose we can write $\omega_k = \sum_{i=1}^{k-1} c_i \omega_i$ for some $c_i \in \mathbb{R}$. Then,

$$\begin{aligned}\omega_1 \wedge \cdots \wedge \omega_k = \omega_1 \wedge \cdots \wedge \omega_{k-1} \wedge \sum_{i=1}^{k-1} c_i \omega_i &= \sum_{i=1}^{k-1} c_i \omega_1 \wedge \cdots \wedge \omega_{k-1} \wedge \omega_i \\ &= \sum_{i=1}^{k-1} (-1)^{k-1-i} c_i \omega_1 \wedge \cdots \wedge \omega_i \wedge \omega_i \wedge \cdots \wedge \omega_{k-1} = \sum_{i=1}^{k-1} 0 = 0\end{aligned}$$

since $\omega_i \wedge \omega_i = 0$ for ω_i a 1-form. Conversely, suppose that $\omega_1, \ldots, \omega_k$ are linearly independent. Take a point $p \in M$ and locally, let V_i be dual vector fields to ω_i. Since $\omega_1, \ldots, \omega_k$ are linearly independent, so too are $V_1, \ldots, V_k$ so

$$\omega_1 \wedge \cdots \wedge \omega_k(V_1, \ldots, V_k) = 1,$$

showing $\omega_1 \wedge \cdots \wedge \omega_k \neq 0$ locally. Since this works in any chart on M, the form cannot be identically zero.

Spring 2014-5. Let $M = \mathbb{R}^2/\mathbb{Z}^2$ be the two dimensional torus, L the line $3x = 7y$ in $\mathbb{R}^2$, and $S = \pi(L) \subset M$ where $\pi : \mathbb{R}^2 \to M$ is the projection map. Find a differential form on M which represents the Poincaré dual of S.

Hint: Expand and solve in terms of the generators of the exterior algebra dx and dy.

Referenced in: Fall 2023-5.

The loop S wraps around 7 times in the x-direction and 3-times in the y-direction. Similar to Fall 2015-4, we have

$$7a + 3b = \int_S adx + bdy = \int_M (Adx + Bdy) \wedge (adx + bdy) = \int_M (Ab - Ba)dx \wedge dy = Ab - Ba,$$

so $B = -7$ and $A = 3$ and the Poincaré dual of S is $3dx - 7dy$.

Spring 2014-6. Let $S^n \subset \mathbb{R}^{n+1}$ be the unit sphere, equipped with the round metric g_S (the restriction of the Euclidean metric on $\mathbb{R}^{n+1}$). Consider also the hyperplane $H = \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$, equipped with the Euclidean metric g_H. Any line passing through the North Pole $p = (0, \ldots, 0, 1)$ and another point $A \in S^n$ will intersect this hyperplane in a point A'. The map

$$\Psi : S^n - \{p\} \to H, \quad \Psi(A) = A'$$

is called the stereographic projection. Show that Ψ is conformal, i.e. for any $x \in S^n - \{p\}$, the bilinear form $(g_S)_x$ is a multiple of the bilinear form $\Psi^*((g_H)_{\Psi(x)})$.

Hint: Metric g_S is $i^*\omega$ for $\omega = dx_1 \otimes dx_1 + \ldots + dx_{n+1} \otimes dx_{n+1}$ the standard metric on $\mathbb{R}^{n+1}$ while g_H is η, the standard metric on $\mathbb{R}^n$. Then, suffices to show $(\Psi^{-1})^* i^* \omega = \mu^2 \eta$ for some function μ which can be done in the $n = 1$ case.

Note that $\Psi^{-1} : \mathbb{R}^n \to S^n - \{p\} \subset \mathbb{R}^n \times \mathbb{R}$ is given by

$$u \mapsto \left(\frac{2u}{|u|^2 + 1}, \frac{|u|^2 - 1}{|u|^2 + 1} \right).$$

Let $i : S^n - \{p\} \to \mathbb{R}^{n+1}$ be the inclusion. The metric g_S is $i^*\omega$ where $\omega = dx_1 \otimes dx_1 + \ldots + dx_{n+1} \otimes dx_{n+1}$ is the standard metric on $\mathbb{R}^{n+1}$. The metric g_H is $\eta = dx_1 \otimes dx_1 + \ldots + dx_n \otimes dx_n$. To show that Ψ is conformal, we would like to show that $i^*\omega$ and $\Psi^*\eta$ differ by a positive function, i.e., that there is some function λ such that $i^*\omega = \lambda^2 \Psi^* \eta$.

It suffices to show $(\Psi^{-1})^* i^* \omega = \mu^2 \eta$ for some function μ, as then $i^*\omega = \Psi^*(\mu^2 \eta) = (\Psi^* \mu^2) \Psi^* \eta = (\Psi^* \mu)^2 \Psi^* \eta$. Now,

$$\begin{aligned}
(\Psi^{-1})^* i^* \omega &= (i \circ \Psi^{-1})^* \omega = (i \circ \Psi^{-1})^* \sum_{i=1}^{n+1} (dx_i \otimes dx_i) \\
&= \left(\sum_{i=1}^{n} d\left(\frac{2u_i}{|u|^2 + 1} \right) \otimes d\left(\frac{2u_i}{|u|^2 + 1} \right) \right) + d\left(\frac{|u|^2 - 1}{|u|^2 + 1} \right) \otimes d\left(\frac{|u|^2 - 1}{|u|^2 + 1} \right).
\end{aligned}$$

Continuing in the $n = 1$ case,

$$\begin{aligned}
&= d\left(\frac{2x}{x^2 + 1} \right) \otimes d\left(\frac{2x}{x^2 + 1} \right) + d\left(\frac{x^2 - 1}{x^2 + 1} \right) \otimes d\left(\frac{x^2 - 1}{x^2 + 1} \right) \\
&= \left(\frac{d}{dx} \left(\frac{2x}{x^2 + 1} \right) \right)^2 dx \otimes dx + \left(\frac{d}{dx} \left(\frac{x^2 - 1}{x^2 + 1} \right) \right)^2 dx \otimes dx \\
&= \frac{4}{(x^2 + 1)^2} dx \otimes dx,
\end{aligned}$$

which is precisely $\mu^2 \eta$ for $\mu(x) = \frac{2}{x^2 + 1}$. Indeed, the calculation for $n > 1$ is far more messy but one can (not easily) see that it similarly gives a positive function times η.

Spring 2014-7. Let X be the wedge sum $S^1 \vee S^1$. Give an example of an irregular covering space $\widetilde{X} \to X$.

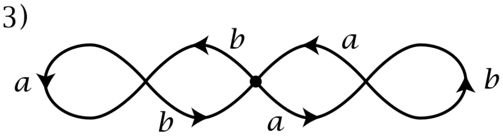
Hint: Subgroup $\langle a^2, b^2, aba^{-1}, bab^{-1} \rangle \subset \langle a, b \rangle$. Corresponding cover has three vertices with a loop on left one, b loop on right one and base point in the middle.

Referenced in: Spring 2024-8.

Recall that regular covering spaces of X are those whose group of deck transformations act transitively on all fibers and correspond to normal subgroups of $\pi_1(X)$. We know $\pi_1(S^1 \vee S^1) = \langle a, b \rangle$ where a corresponds to a loop that goes counterclockwise around the first copy of S^1 and b to the second copy of S^1. I.e., a covering space $\widetilde{X}$ is regular if and only if for any v, w vertices of $\widetilde{X}$, there is a symmetry σ such that $\sigma(v) = w$.

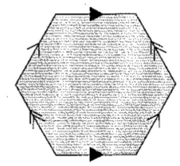
The subgroup $\langle a^2, b^2, aba^{-1}, bab^{-1} \rangle \subset \langle a, b \rangle$ is not normal and the corresponding covering space is shown below where it is clear that there is no symmetry σ that takes the left vertex to the middle one since the left vertex has a loop while the middle does not.

(3)



$\langle a^2, b^2, aba^{-1}, bab^{-1} \rangle$

Spring 2014-8. For $n \geqslant 2$, let X_n be the space obtained from a regular $(2n)$-gon by identifying the opposite sides with parallel orientations. For example, X_3 is



The above description produces a cell decomposition of X_n.

(a) Write down the associated cellular chain complex.
(b) Show that X_n is a surface, and find its genus.

Hint: Split into even and odd case, odd has two 0-cells. Classification of compact, orientable 2-manifolds. $g = n/2$ or $(n-1)/2$ for n even, odd, respectively.

(a) If n is even, we have only one 0-cell, while if n is odd, we have two 0-cells. In both cases, we have n 1-cells, $a_1, \ldots, a_n$, corresponding to the pairs of opposite sides, and one 2-cell, F, attached via $a_1 a_2 \cdots a_n a_1^{-1} a_2^{-1} \cdots a_n^{-1}$. Thus, the cell chain complex is

$$0 \to \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z}^n \xrightarrow{\partial_1} \mathbb{Z} \to 0$$

for n even. In this case, $\partial_2(F) = a_1 + \ldots + a_n - a_1 - \ldots - a_n = 0$ and $\partial_1(a_i) = v - v = 0$ for all i so both maps are 0. And for n odd, we have

$$0 \to \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z}^n \xrightarrow{\partial_1} \mathbb{Z}^2 \to 0$$

where we again have $\partial_2 = 0$ but now, $\partial_1(a_i) = (-1)^i v + (-1)^{i+1} w$ for v, w the two generators of $\mathbb{Z}^2 = C_0$. I.e., $\partial_1(z_1, \ldots, z_n) = (-z_1 + z_2 - \ldots + z_{n-1} - z_n, z_1 - z_2 + \ldots - z_{n-1} + z_n)$.

(b) First, note that each point in the $(2n)$-gon has a neighborhood homeomorphic to $\mathbb{R}^2$. Each point on an edge has a neighborhood homeomorphic to the closed half plane, but since the edge is identified (glued) with another edge, after identification, the neighborhood is homeomorphic to $\mathbb{R}^2$. Similarly, each point on a vertex has a neighborhood homeomorphic to $\mathbb{R}^2$ after the identification.

Clearly the polygon itself is compact so X_n, as the quotient space of a compact space, is also compact. Based on the cell complex described in part (a), we can see that $H_2(X_n) = \ker(\partial_2) = \mathbb{Z}$ so X_n is orientable. So by the classification of compact orientable 2-manifolds, X_n is homeomorphic to M_g, a torus of genus g for some g.

To find g, we calculate the Euler characteristic of X_n since we know that $\chi(M_g) = 2 - 2g$. For n even, $\chi(X_n) = 1 - n + 1 = 2 - n$ and for n odd, $\chi(X_n) = 1 - n + 2 = 3 - n$. Hence, for n even, $g = \frac{n}{2}$ while for n odd, $g = \frac{n-1}{2}$.

Spring 2014-9. (a) Consider the space Y obtained from $S^2 \times [0,1]$ by identifying $(x, 0)$ with $(-x, 0)$ and also identifying $(x, 1)$ with $(-x, 1)$, for all $x \in S^2$. Show that Y is homeomorphic to the connected sum $\mathbb{RP}^3 \# \mathbb{RP}^3$.
(b) Show that $S^2 \times S^1$ is a double cover of the connected sum $\mathbb{RP}^3 \# \mathbb{RP}^3$.

Hint: Removing a 3-cell from $\mathbb{RP}^3$ gives $\mathbb{RP}^2$ so $\mathbb{RP}^3 \# \mathbb{RP}^3$ is two copies of $\mathbb{RP}^2$ glued at the ends of a 2-tube $S^2 \times [0,1]$ which is exactly what Y is. Standard double cover $S^2 \to \mathbb{RP}^2$ at the ends and then $S^2 \times (0, 0.5)$ covers it once and $S^2 \times (0.5, 1)$ covers it a second time.

(a) Note that $S^2/\sim = \mathbb{RP}^2$ where $x \sim -x$ for all $x \in S^2$ since this is exactly the antipodal identification. Thus, Y is the union of $S^2 \times (0,1)$ and $\mathbb{RP}^2 \times \{0, 1\}$. On the other hand, consider the CW-structure on $\mathbb{RP}^3$

with one k-cell for each $k = 0, 1, 2, 3$. Then, $\mathbb{RP}^3 \# \mathbb{RP}^3$ can be formed by removing the 3-cell of each copy of $\mathbb{RP}^3$ and gluing the underlying 2-skeletons (which is exactly $\mathbb{RP}^2$) to each end of the 3-cylinder $S^2 \times [0, 1]$ via the covering map $S^2 \to \mathbb{RP}^2$. One can easily see that this is the same as Y as described above.

(b) We use part (a) to identify $\mathbb{RP}^3 \# \mathbb{RP}^3$ with $S^2 \times [0, 1]/\sim$. Let $S^1 = [0, 1]/\sim_0$ where $0 \sim_0 1$. Then, define

$$p : S^2 \times ([0, 1]/\sim_0) \to (S^2 \times [0, 1])/\sim, \quad p(x, t) = \begin{cases} (x, 2t) & 0 \leqslant t \leqslant 0.5, \\ (x, 2 - 2t) & 0.5 \leqslant t \leqslant 1. \end{cases}$$

Note that this is smooth/well-defined since $(x, 0) = (x, 1) \in S^2 \times ([0, 1]/\sim_0)$ both get sent to $(x, 0) \in (S^2 \times [0, 1])/\sim$. For $(x, 0) \in (S^2 \times [0, 1])/\sim$, we have the standard double cover $S^2 \times \{0\} \to \mathbb{RP}^2 \times \{0\}$ and similarly $(x, 1)$ is covered by $S^2 \times \{0.5\} \to \mathbb{RP}^2 \times \{1\}$. For any other (x, t), we have both $(x, \frac{t}{2})$ and $(x, 1 - \frac{t}{2})$ getting sent to (x, t) so indeed, this is a double cover.

Spring 2014-10. Let X be a topological space. Define the suspension $S(X)$ to be the space obtained from $X \times [0, 1]$ by contracting $X \times \{0\}$ to a point, and contracting $X \times \{1\}$ to another point. Describe the relation between the homology groups of X and $S(X)$.

Hint: $\widetilde{H}_k(S(X)) = \widetilde{H}_{k-1}(X)$ for all k. Use Mayer-Vietoris.

This is exactly Fall 2020-6.

Fall 2013

Fall 2013-1. Let $f : M \to N$ be a nonsingular smooth map between connected manifolds of the same dimension. Answer the following questions with a proof or counter-example.

(a) Is f necessarily injective or surjective?
(b) Is f necessarily a covering map when N is compact?
(c) Is f necessarily an open map?
(c) Is f necessarily a closed map?

Hint: All but (c) false, consider $t \mapsto e^{it}$ and $(0, 1) \hookrightarrow \mathbb{R}$ and $(-0.1, 1.1) \to S^1$. Local homeomorphisms are open maps, just restrict to neighborhood.

Referenced in: Fall 2014-1.

(a) Both are false. $f : \mathbb{R} \to S^1, f(t) = e^{it}$ is not injective and $i : (0, 1) \hookrightarrow \mathbb{R}$ is not surjective but both are local diffeomorphisms between connected manifolds of the same dimension.

(b) False. $f : (-0.1, 1.1) \to S^1, f(x) = e^{2\pi i x}$ is not a covering map since $f^{-1}(1) = \{0, 1\}$ while $f^{-1}(-1) = \{0.5\}$.

(c) True. Since f is a local diffeomorphism (nonsingular and $\dim(M) = \dim(N)$), it is a local homeomorphism. Let $V \subset M$ be a nonempty open subset and let $y \in f(V)$. Choose some $x \in f^{-1}(y) \cap V$. Since f is a local homeomorphism, find a neighborhood $U \ni x$ such that $f|_U : U \to f(U)$ is a homeomorphism onto $f(U) \subset N$ an open subset. Then, $f|_U$ is an open map so $f|_U(U \cap V) = f(U \cap V) \subset f(V)$ is an open neighborhood of $y \in f(V)$ so $f(V)$ is an open set and f is an open map.

(d) False. $i : (0, 1) \hookrightarrow \mathbb{R}$ has $i(0, 1) = (0, 1)$ which is not closed in $\mathbb{R}$ while $(0, 1)$ is trivially closed in $(0, 1)$.

Fall 2013-2. Let M be a connected compact manifold with non-empty boundary ∂M. Show that M does not retract onto ∂M.

Hint: Use $\mathbb{Z}/2\mathbb{Z}$ coefficients in order to apply Lefschetz duality to simplify the long exact sequence for the pair $(M, \partial M)$. Get $\ker(H_{n-1}(\partial M) \to H_{n-1}(M)) \cong \mathbb{Z}/2\mathbb{Z}$, contradicting the fact that it should be an injection if $r : M \to \partial M$ is a retraction.

This is exactly Spring 2023-4.

Fall 2013-3. Let $M, N \subset \mathbb{R}^{p+1}$ be two compact, smooth, oriented submanifolds of dimensions m and n, respectively, such that $m + n = p$. Suppose that $M \cap N = \varnothing$. Consider the linking map

$$\lambda : M \times N \to S^p, \quad \lambda(x, y) = \frac{x - y}{||x - y||}.$$

The degree of λ is called the linking number $l(M, N)$.

(a) Show that $l(M, N) = (-1)^{(m+1)(n+1)} l(N, M)$.
(b) Show that if M is the boundary of an oriented submanifold $W \subset \mathbb{R}^{p+1}$ disjoint from N, then $l(M, N) = 0$.

Hint: $l(N, M) = \deg(\lambda'), \lambda' = A \circ \lambda_M \circ S : N \times M \to S^p$. Define $F(w, n) = \frac{w-n}{||w-n||}$ so that $\lambda = \partial F = F \circ i_M$ so $d\lambda^*\omega = dF^*\omega$ and use Stokes' to integrate $\deg(\lambda) \int_{S^p} \omega$ for any volume form ω.

This is exactly Fall 2020-5.

Fall 2013-4. Let ω be a 1-form on a connected manifold M. Show that ω is exact, i.e., $\omega = df$ for some function f, if and only if for all piecewise smooth closed curves $c : S^1 \to M$ it follows that $\int_c \omega = 0$.

Hint: Stokes'. Define $g(x) = \int_{\gamma_x} \omega$ where γ_x is path from x_0 to x. Well-defined since integrating any loop gives 0 by assumption. Then $dg = \omega$.

Referenced in: Spring 2013-2.

This is essentially the same as Spring 2019-5. The reverse direction there is purely stronger while for the forward direction, if $\omega = df$ is exact, we have

$$\int_c \omega = \int_c df = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} c^* df = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} d(f \circ c) = \sum_{i=1}^n f(c(t_i)) - f(c(t_{i-1})) = f(c(t_n)) - f(c(t_0)) = 0.$$

Fall 2013-5. Let ω be a smooth, nowhere vanishing 1-form on a three dimensional smooth manifold M^3.

(a) Show that $\ker(\omega)$ is an integrable distribution on M if and only if $\omega \wedge d\omega = 0$.
(b) Give an example of a codimension one distribution on $\mathbb{R}^3$ that is not integrable.

Hint: $\ker(\omega) = \text{span}(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^{m-1}}), \omega|_U = f_U dx^m$. Set $\alpha|_U = \frac{df_U}{f_U}$ and put together with partition of unity. For backward direction, use Frobenius's theorem with $d\omega(X, Y) = -\omega([X, Y])$ to show $[X, Y] \in \ker(\omega)$. Take $\omega = -ydx + xdy + dz$.

(a) This is exactly (a) $\iff$ (b) in Fall 2022-4.

(b) Let $\omega = -ydx + xdy + dz$. Then

$$\omega \wedge d\omega = (-ydx + xdy + dz) \wedge (2dx \wedge dy) = 2dx \wedge dy \wedge dz \neq 0,$$

so $\ker(\omega)$ is not integrable by part (a).

Fall 2013-6. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function.

(a) Define the gradient ∇f as a vector field dual to the differential df.
(b) Define the Hessian $\text{Hess}f(X,Y)$ as a symmetric $(0,2)$-tensor.
(c) If the usual Euclidean inner product between tangent vectors in $T_p\mathbb{R}^n$ is denoted $g(X,Y) = X \cdot Y$, show that

$$\text{Hess}f(X,Y) = \frac{1}{2} L_{\nabla f} g(X,Y).$$

Here $L_Z g$ is the Lie derivative of g in the direction of Z.

Hint: $\nabla f = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_i}$, $H_f = \sum_{1\leqslant i,j\leqslant n} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \otimes dx_j$. Expand out with $g = \sum_{i=1}^n dx_i \otimes dx_i$. Cartan's magic formula.

(a) We know that the dual of dx_i is $(dx_i)^* = \frac{\partial}{\partial x_i}$. Thus, since $df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$, we have

$$\nabla f = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_i}.$$

(b) We will write $H_f(X,Y)$ for $\text{Hess}f(X,Y)$ for simplicity. We have

$$H_f = \sum_{1\leqslant i,j\leqslant n} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \otimes dx_j.$$

Note that this is symmetric since mixed partials commute $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$.

(c) Note that $g = \sum_{i=1}^n dx_i \otimes dx_i$. Now,

$$\begin{aligned} L_{\nabla f} g &= L_{\nabla f}\left(\sum_{i=1}^n dx_i \otimes dx_i\right) = \sum_{i=1}^n L_{\nabla f}(dx_i \otimes dx_i) \\ &= \sum_{i=1}^n (L_{\nabla f} dx_i) \otimes dx_i + dx_i \otimes (L_{\nabla f} dx_i). \end{aligned}$$

Then, by Cartan's magic formula,

$$L_{\nabla f} dx_i = i_{\nabla f} d(dx_i) + d i_{\nabla f} dx_i = 0 + d((\nabla f)(x_i)) = d\left(\frac{\partial f}{\partial x_i}\right) = \sum_{j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j} dx_j.$$

So we have,

$$\begin{aligned} L_{\nabla f} g &= \sum_{i=1}^n \left(\sum_{j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j} dx_j\right) \otimes dx_i + \sum_{i=1}^n dx_i \otimes \left(\sum_{j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j} dx_j\right) \\ &= \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j} dx_j \otimes dx_i + \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \otimes dx_j = 2H_f, \end{aligned}$$

using the equality of mixed partials and symmetricity of tensors.

Fall 2013-7. Let $M = T^2 - D^2$ be the complement of a disk inside the two-torus. Determine all connected surfaces that can be described as 3-fold covers of M.

Hint: Graphs with 3 vertices of degree 4 each. Consider the edges a and b. Seven (thirteen) total, four normal.

Note that M is homotopy equivalent to a punctured torus which we know deformation retracts onto $S^1 \vee S^1$. Thus, this is essentially the same as Fall 2020-8. Note that if we kept track of base point, we would have thirteen 3-sheeted covers instead because the third, fifth, and sixth examples have no symmetry so split into three examples each. Then, to get a covering of M, we would attached three 2-cells to each of these graphs via the boundary words $aba^{-1}b^{-1}$, one starting/ending at each vertex and remove a D^2 from each 2-cell.

Fall 2013-8. Let $n > 0$ be an integer and let A be an abelian group with a finite presentation by generators and relations. Show that there exists a topological space X with $H_n(X) \cong A$.

Hint: Fundamental theorem of finitely generated abelian groups. Disjoint union of spaces gives direct sum of homology groups. Attach $(n+1)$-cell to S^n by a degree m map to get $\mathbb{Z}/m\mathbb{Z}$.

Referenced in: Fall 2010-7.

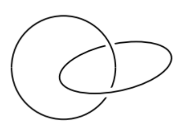
Note that finitely presented abelian groups are exactly the finitely generated ones and we know that any such group can be uniquely written as

$$A \cong \mathbb{Z}^r \oplus \mathbb{Z}/m_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_k\mathbb{Z},$$

for some $r \geqslant 0, m_1 \mid m_2 \mid \cdots \mid m_k$ natural numbers. Let $X = S^n$ for which we know $H_n(X) = \mathbb{Z}$. For $m \in \mathbb{N}$, let X_m be the space with CW structure obtained by attaching an $(n+1)$-cell to S^n by a degree m map $f_m : \partial D^{n+1} = S^n \to S^n$. (We know such a degree m map exists say by Fall 2022-3.) It is clear that $H_n(X_m) = \mathbb{Z}/\mathrm{im}(f_m) = \mathbb{Z}/m\mathbb{Z}$. Then, using the fact that $H_*(Y \sqcup Z) \cong H_*(Y) \oplus H_*(Z)$, we get

$$H_*\left(\bigsqcup_{i=1}^{r} X \sqcup X_{m_1} \sqcup \cdots \sqcup X_{m_k}\right) \cong \mathbb{Z}^r \oplus \mathbb{Z}/m_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_k\mathbb{Z} = A.$$

Fall 2013-9. Let $H \subset S^3$ be the Hopf link, shown in the figure



Compute the fundamental group and the homology groups of the complement $S^3 - H$.

Hint: $\pi_1(S^3 - H) = \mathbb{Z}^2$. $H_*(S^3 - H) = \mathbb{Z}_{(2)} \oplus \mathbb{Z}^2_{(1)} \oplus \mathbb{Z}_{(0)}$. Homotopy equivalent to torus T^2.

We explained in Fall 2017-10 how $X = S^3 - H$ is homotopy equivalent to T^2, the standard torus so has fundamental group $\pi_1(X) = \mathbb{Z}^2$. The homology groups of the torus are also very easy to see by inspection (or Künneth's formula):

$$H_k(X) = \begin{cases} \mathbb{Z} & k = 0, 2, \\ \mathbb{Z}^2 & k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Fall 2013-10. Let $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ be the group of quaternions, with relations $i^2 = j^2 = -1, ij = -ji = k$. The multiplicative group $\mathbb{H}^* = \mathbb{H} - \{0\}$ acts on $\mathbb{H}^n - \{0\}$ by left multiplication. The quotient $\mathbb{HP}^{n-1} = (\mathbb{H}^n - \{0\})/\mathbb{H}^*$ is called the quaternionic projective space. Calculate its homology groups.

Hint: One cell in each dimension $4k$ for $0 \leqslant k \leqslant n$. So $H_*(\mathbb{HP}^n) = \mathbb{Z}_{(0)} \oplus \mathbb{Z}_{(4)} \oplus \cdots \oplus \mathbb{Z}_{(4n)}$.

We give $\mathbb{HP}^n$ a cell structure with one $4k$-cell for each $0 \leqslant k \leqslant n$ by induction. Clearly $\mathbb{HP}^0$ is just a single point, i.e., a single 0-cell. Then, given such a cell structure on $\mathbb{HP}^{n-1}$, we attach a $(4n)$-cell via

$$\phi_n : \partial D^{4n} = S^{4n-1} \to \mathbb{HP}^{n-1}, \quad (a_0, \dots, a_{n-1}) \mapsto [a_0 : \cdots : a_{n-1}],$$

where we are viewing $S^{4n-1} \subset \mathbb{R}^{4n} \cong \mathbb{H}^n$. Then, we can see that $D^{4n} \cup_{\phi_n} \mathbb{HP}^{n-1} \cong \mathbb{HP}^n$, with homeomorphism given by

$$f : D^{4n} \cup_{\phi_n} \mathbb{HP}^{n-1} \to \mathbb{HP}^n, \qquad \begin{cases} [a_0 : \cdots : a_{n-1}] \mapsto [a_1 : \cdots : a_{n-1} : 0] & \text{on } \mathbb{HP}^{n-1}, \\ (a_0, \dots, a_{n-1}) \mapsto \left[a_0 : \cdots : a_{n-1} : \sqrt{1 - \sum_{i=0}^{n-1} |a_i|^2}\right] & \text{on } D^{4n}. \end{cases}$$

On the boundary ∂D^{4n}, we have $\sum_{i=0}^{n-1} |a_i|^2 = 1$ so indeed this is well-defined and continuous. Moreover, it is not hard to see that this is a bijection with continuous inverse. The corresponding cell complex thus has $\mathbb{Z}$ in degree $4k$ for $0 \leqslant k \leqslant n$ and 0 elsewhere. So all the boundary maps are 0 and the homology is just

$$H_i(\mathbb{HP}^n) = \begin{cases} \mathbb{Z} & i = 4k, 0 \leqslant k \leqslant n, \\ 0 & \text{otherwise.} \end{cases}$$

Spring 2013

Spring 2013-1. Let $\mathrm{Mat}_{m \times n}(\mathbb{R})$ be the space of $m \times n$ matrices with real valued entries.

(a) Show that the subset $S \subset \mathrm{Mat}_{m \times n}(\mathbb{R})$ of rank 1 matrices forms a submanifold of dimension $m + n - 1$.
(b) Show that the subset $T \subset \mathrm{Mat}_{m \times n}(\mathbb{R})$ of rank k matrices form a submanifold of dimension $k(m + n - k)$.

Hint: $\begin{pmatrix} I_k & 0 \\ -CA^{-1} & I_k \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix}$ so consider $f : N \to M_{n-k}$ given by $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto D - CA^{-1}B$. Preimage theorem.

This is exactly Fall 2018-2 with $m \times n$ instead of $n \times n$ (but this only changes the dimensions throughout).

Spring 2013-2. Let M be a smooth manifold and $\omega \in \Omega^1(M)$ a smooth 1-form.

(a) Define the line integral
$$\int_c \omega$$
along piecewise smooth curves $c : [0, 1] \to M$.
(b) Show that $\omega = df$ for a smooth function $f : M \to \mathbb{R}$ if and only if $\int_c \omega = 0$ for all closed curves $c : [0, 1] \to M$, i.e., $c(0) = c(1)$.

Hint: Define as sum of integrals on each piece of $\gamma_i^* \omega$. Stokes'. Define $g(x) = \int_{\gamma_x} \omega$ where γ_x is path from x_0 to x. Well-defined since integrating any loop gives 0 by assumption. Then $dg = \omega$.

(a) For $c : [0, 1] \to M$ piecewise smooth, with each $\gamma_i = c|_{[t_{i-1}, t_i]} : [t_{i-1}, t_i] \to M$ smooth for $0 \leqslant t_0 \leqslant t_1 \leqslant \ldots \leqslant t_n = 1$, we define

$$\int_c \omega = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \gamma_i^* \omega.$$

(b) This is exactly Fall 2013-4.

Spring 2013-3. Let $S_1, S_2 \subset M$ be smooth embedded submanifolds.

(a) Define what it means for S_1, S_2 to be transversal.
(b) Show that if $S_1, S_2 \subset M$ are transversal then $S_1 \cap S_2 \subset M$ is a smooth embedded submanifold of dimension $\dim S_1 + \dim S_2 - \dim M$.

Hint: $T_xS_1 + T_xS_2 = T_xM$ for all $x \in S_1 \cap S_2$. Work locally with a slice chart and compose with the inclusion. Show 0 is still a regular value using transversality, then preimage theorem.

(a) S_1 and S_2 are transversal if for all $x \in S_1 \cap S_2$, we have

$$T_xS_1 + T_xS_2 = T_xM.$$

This is equivalent to saying that the inclusion map $i : S_1 \hookrightarrow M$ is transverse to S_2, i.e., for each $x \in S_1$ such that $i(x) \in S_2$, we have

$$di_x(T_xS_1) + T_{i(x)}S_2 = T_{i(x)}M.$$

(b) Write $\dim M = m, \dim S_1 = s_1, \dim S_2 = s_2$. Let $i : S_1 \to M$ be the embedding of S_1 into M and let $p \in S_1 \cap S_2$. Since S_2 is an embedded submanifold of M of codimension $m - s_2$, we can find an open neighborhood $U \subset M$ of p and $m - s_2$ independent functions $g_1, \ldots, g_{m-s_2} : U \to \mathbb{R}$ such that $U \cap S_2 = \{q \in U \mid g_1(q) = \cdots = g_{m-s_2}(q) = 0\}$. Thus, we have

$$U \cap S_1 \cap S_2 = \{q \in U \cap S_1 \mid g_1 \circ i(q) = \cdots = g_{m-s_2} \circ i(q) = 0\}.$$

We see that $g : U \to \mathbb{R}^{m-s_2}$ defined by $g = (g_1, \ldots, g_{m-s_2})$ has 0 as a regular value with $g^{-1}(0) = U \cap S_2$. Then, consider $g \circ i : U \cap S_1 \to \mathbb{R}^{m-s_2}$. Note that $(g \circ i)^{-1}(0) = U \cap S_1 \cap S_2$ and we claim that 0 is a regular value of $g \circ i$. For this, let $q \in (g \circ i)^{-1}(0)$ so that $d(g \circ i)_q = dg_q \circ di_q$ where $di_q : T_q(U \cap S_1) \to T_qU$ and $dg_q : T_qU \to \mathbb{R}^{m-s_2}$. Since i is transversal to S_2, we have $T_qU = di_q(T_q(U \cap S_1)) + T_q(U \cap S_2)$. By construction, dg_q is zero on $T_q(U \cap S_2)$ since g is constant on $U \cap S_2$. Thus, dg_q is surjective even when restricted to $di_q(T_q(U \cap S_1))$ which exactly implies that $dg_q \circ di_q$ is surjective so indeed 0 is a regular value of $g \circ i$.

Thus, by the preimage theorem, $U \cap S_1 \cap S_2$ is a submanifold of $U \cap S_1$ of codimension $m - s_2$, i.e., of dimension $s_1 - (m - s_2) = s_1 + s_2 - m$. Since a manifold structure is determined locally, we conclude that $S_1 \cap S_2$ is a submanifold of S_1 of dimension $s_1 + s_2 - m$.

Spring 2013-4. Let $C \subset M$ be given as $F^{-1}(c)$ where $F = (F^1, \ldots, F^k) : M \to \mathbb{R}^k$ is smooth and $c \in \mathbb{R}^k$ is a regular value for F. If $f : M \to \mathbb{R}$ is smooth, show that its restriction $f|_C$ to a submanifold $C \subset M$ has a critical point at $p \in C$ if and only if there exist constants $\lambda_1, \ldots, \lambda_k$ such that

$$df_p = \sum \lambda_i dF_p^i$$

where $dg_p : T_pM \to \mathbb{R}$ denotes the differential at p of a smooth function g.

Hint: F is constant on C so $dF_p^i = 0$ for all i and any $p \in C$. df_p can only be nonzero on directions normal to C for which dF_p^i form a basis.

Let $i : C \hookrightarrow M$ be the inclusion. A point $p \in C$ is a critical point of $f|_C$ if and only if $d(f|_C)_p : T_pC \to T_{f(p)}\mathbb{R} = \mathbb{R}$ is not surjective. But $d(f|_C)_p = d(f \circ i)_p = df_p \circ di_p$, and since $\mathbb{R}$ is 1-dimensional, this is surjective if and only if it is nonzero. If $df_p = \sum_{j=1}^k \lambda_j (dF^j)_p$, then we have

$$df_p \circ di_p = \sum_{j=1}^k \lambda_j (dF^j)_p \circ di_p = \sum_{j=1}^k \lambda_j d(F^j \circ i)_p = \sum_{j=1}^k \lambda_j \cdot 0 = 0,$$

since each $F^j \circ i : C \to \mathbb{R}^k$ is just a constant function $C \mapsto c_j$ by assumption. So certainly if df_p is of this form, p will be a critical point. Conversely, suppose that $p \in C$ is a critical point of $f|_C$ so $df_p \circ di_p = 0$. Write $T_pM = T_pC \oplus V_p$ where V_p is the orthogonal complement of T_pC in T_pM. Then, note that $di_p : T_pC \to T_pC \oplus V_p$ is an injection into the first factor.

Thus, $df_p(T_pC) = 0$ so df_p is nonzero only in directions normal to C. By construction, we know that $dF_p^1, \ldots, dF_p^k$ form a (dual) basis for V_p. Thus, we can find constants $\lambda_1, \ldots, \lambda_k$ such that $df_p = \sum_{j=1}^k \lambda_j dF_p^j$ as desired.

Spring 2013-5. Let M be a smooth, orientable, compact manifold with boundary ∂M. Show that there is no (smooth) retract $r : M \to \partial M$.

Hint: Use $\mathbb{Z}/2\mathbb{Z}$ coefficients in order to apply Lefschetz duality to simplify the long exact sequence for the pair $(M, \partial M)$. Get $\ker(H_{n-1}(\partial M) \to H_{n-1}(M)) \cong \mathbb{Z}/2\mathbb{Z}$, contradicting the fact that it should be an injection if $r : M \to \partial M$ is a retraction.

This is exactly Spring 2023-4.

Spring 2013-6. Let $A \in GL_{n+1}(\mathbb{C})$.

(a) Show that A defines a smooth map $A : \mathbb{CP}^n \to \mathbb{CP}^n$.
(b) Show that the fixed points of $A : \mathbb{CP}^n \to \mathbb{CP}^n$ correspond to eigenvectors for the original matrix.
(c) Show that $A : \mathbb{CP}^n \to \mathbb{CP}^n$ is a Lefschetz map if the eigenvalues of A all have multiplicity 1.
(d) Show that the Lefschetz number of $A : \mathbb{CP}^n \to \mathbb{CP}^n$ is $n + 1$. Hint: You are allowed to use that $Gl_{n+1}(\mathbb{C})$ is connected.

Hint: $(d\phi_A)_x$ has no fixed points for any eigenvector x of A. Use local coordinates. Since we can homotope anywhere by connectedness, use $A = \mathrm{diag}(1, 2, \ldots, n+1)$ to use part (c) so Lefschetz number is sum of local Lefschetz numbers which are all $+1$ so answer is $n + 1$.

Referenced in: Spring 2024-9.

(a) Define $A : \mathbb{CP}^n \to \mathbb{CP}^n$ by

$$[z_0 : \cdots : z_n] = [(Az)_1 : \cdots : (Az)_n]$$

where we treat $z = (z_0, \ldots, z_n)^T$ as a column vector in $\mathbb{C}^{n+1}$ and we write v_i for the ith component of a vector v. Now, since A is invertible, it descends to a map $\mathbb{C}^{n+1} - \{0\} \to \mathbb{C}^{n+1} - \{0\}$. Then, note that $A(\lambda z) = \lambda A(z)$ so that $A(\lambda z) \sim A(z)$ in $\mathbb{CP}^n$, showing that A indeed factors through the map $\mathbb{CP}^n \to \mathbb{CP}^n$ shown above which is thus well-defined. Finally, smoothness is merely inherited from $A \in GL_{n+1}(\mathbb{C})$ being linear and thus a smooth map from $\mathbb{C}^{n+1}$ to $\mathbb{C}^{n+1}$.

(b), (c), and (d) This is exactly Spring 2022-10.

Spring 2013-7. Let $F : S^n \to S^n$ be a continuous map.

(a) Define the degree $\deg F$ of F and show that when F is smooth

$$\deg F \int_{S^n} \omega = \int_{S^n} F^* \omega$$

for all $\omega \in \Omega^n(S^n)$.
(b) Show that if F has no fixed points then $\deg F = (-1)^{n+1}$.

Hint: $\int_{S^n} F^* \omega = \int_{F_* S^n} \omega$ since we can view S^n as an n-cycle in the domain S^n. Lefschetz trace fixed point formula.

Referenced in: Spring 2011-3, Spring 2010-8.

(a) Note that $H_n(S^n) = \mathbb{Z}$ so $F : S^n \to S^n$ induces a map $F_* : \mathbb{Z} \to \mathbb{Z}$ on nth homology. Such a(n additive) group homomorphism is precisely multiplication by some $\alpha \in \mathbb{Z}$ and we define $\deg(F) = \alpha$. I.e., $\deg(F) = F_*[1]$ on top homology. If we view S^n as an n-cycle in the domain S^n, we have

$$\int_{S^n} F^*\omega = \int_{F_* S^n} \omega = \deg F \int_{S^n} \omega,$$

since $F_*(S^n)$ is a $(\deg F)$-fold cover of S^n.

(b) This is exactly part (a) of Spring 2021-2.

Spring 2013-8. Let $f : S^{n-1} \to S^{n-1}$ be a continuous map and D^n the disk with $\partial D^n = S^{n-1}$.

(a) Define the adjunction space $D^n \cup_f D^n$.
(b) Let $\deg f = k$ and compute the homology groups $H_p(D^n \cup_f D^n; \mathbb{Z})$ for $p = 0, 1, \ldots$.
(c) Assume that f is a homeomorphism, show that $D^n \cup_f D^n$ is homeomorphic to S^n.

Hint: Cell structure with one 0-cell, one $(n-1)$-cell, and two n-cells. Think of $S^n = D^n \cup_{\mathrm{id}} D^n$.

(a) $D^n \cup_f D^n = (D^n \sqcup D^n)/\sim$ where $x \sim f(x)$ for all $x \in \partial D^n = S^{n-1}$. I.e., we have two copies of D^n and we glue their boundaries together via the function f.

(b) Let $X = D^n \cup_f D^n$. Then X has a CW structure with one 0 cell v, an $(n-1)$-cell e attached to v to make S^{n-1}, and two n-cells f_1 and f_2, one of which is attached via the identity map (in order to make the first copy of D^n), and the other is attached via the degree k map $f : S^{n-1} \to S^{n-1}$ (this is the other D^n glued to the first by f). So X has cell chain complex

$$0 \to \mathbb{Z}^2 \xrightarrow{\partial} \mathbb{Z} \to 0 \to \cdots \to 0 \to \mathbb{Z} \to 0,$$

where the nonzero terms only appear in degree $n, n-1$, and 0. (Note that $n > 1$ so $n-1 > 0$ and there is no overlap to worry about.) The single nonzero map ∂ is given by $(a, b) \mapsto a + kb$ by definition of the attaching maps. Hence, we easily compute

$$H_k(X) = \begin{cases} \mathbb{Z} & k = 0, n, \\ 0 & \text{otherwise.} \end{cases}$$

(c) We can think of S^n as $D^n \cup_{\mathrm{id}} D^n$ and view $D^n \subset \mathbb{R}^n$ as the unit n-ball centered at the origin. Then define

$$g : D^n \cup_{\mathrm{id}} D^n \to D^n \cup_f D^n, \quad \begin{cases} x \mapsto x & \text{between the first copies of } D^n, \\ x \mapsto ||x|| f(\frac{x}{||x||}) & \text{between the second copies of } D^n. \end{cases}$$

On the boundary $S^{n-1} \subset D^n$, we note that these two cases agree because $||x|| = 1$ in the first copy of D^n in $D^n \cup_f D^n$ is identified with $f(x)$ in the second copy of D^n by definition. Moreover, this is continuous bijection from a compact space to a Hausdorff space so is a homeomorphism.

Spring 2013-9. Let $F : M \to N$ be a finite covering map between closed manifolds. Either prove or find counter examples to the following questions.

(a) Do M and N have the same fundamental groups?
(b) Do M and N have the same de Rham cohomology groups?
(c) When M is simply connected, do M and N have the same singular homology groups?

Hint: All false with counterexample $p : S^2 \to \mathbb{RP}^2$.

All three are false with the same counterexample $p : S^2 \to \mathbb{RP}^2$. We know $\pi_1(S^2) = 0$ while $\pi_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$. Also $H^2_{dR}(S^2) = \mathbb{R}$ while $H^2_{dR}(\mathbb{RP}^2) = 0$. Finally, $H_2(S^2) = \mathbb{Z}$ while $H_2(\mathbb{RP}^2) = 0$.

Spring 2013-10. Let $A \subset X$ be a subspace of a topological space. Define the relative singular homology groups $H_p(X, A)$ and show that there is a long exact sequence

$$\cdots \to H_p(A) \to H_p(X) \to H_p(X, A) \to H_{p-1}(A) \to \cdots.$$

Hint: $C_n(X, A) = C_n(X)/C_n(A)$. Show we have short exact sequence of chain complexes which induces two of the three maps on homology and the last is the connecting map from the snake lemma.

On the level of chain complexes, we have $C_n(X, A) = C_n(X)/C_n(A)$ where the latter forms a chain complex with boundary operator $\partial[\Delta] = [\partial\Delta]$. This is well-defined because if $\Delta \in C_n(X)$ has $[\Delta] = 0$, then $\Delta \in C_n(A)$ so $\partial\Delta \in C_{n-1}(A)$ and $[\partial\Delta] = 0$. Moreover, it is clear that $\partial \circ \partial = 0$.

Now, we claim that

$$0 \to C_\bullet(A) \to C_\bullet(X) \to C_\bullet(X, A) \to 0$$

forms a short exact sequence of chain complexes. For each n, we have a short exact sequence of abelian groups

$$0 \to C_n(A) \to C_n(X) \to C_n(X)/C_n(A) \to 0$$

which by the above comment is the same as

$$0 \to C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{j} C_n(X, A) \to 0.$$

It is clear that $i \circ \partial = \partial \circ i$ by definition. Moreover $j \circ \partial(\Delta) = j(\partial\Delta) = [\partial\Delta] = \partial[\Delta] = \partial \circ j(\Delta)$ so we also have $j \circ \partial = \partial \circ j$. Thus, i and j are indeed maps of chain complexes, proving the claim. Then, the maps $H_n(A) \to H_n(X)$ and $H_n(X) \to H_n(X, A)$ are simply induced by i and j respectively for each n. Finally, the map $H_n(X, A) \to H_{n-1}(A)$ is the connecting map obtained from applying the snake lemma (say as in part (b) of Fall 2022-6).

Fall 2012

Fall 2012-1. (a) Show that the Lie group $SL_2(\mathbb{R}) = \{A \in M_{2\times 2}(\mathbb{R}) \mid \det(A) = 1\}$ is diffeomorphic to $S^1 \times \mathbb{R}^2$.

(b) Show that the Lie group $SL_2(\mathbb{C}) = \{A \in M_{2\times 2}(\mathbb{C}) \mid \det(A) = 1\}$ is diffeomorphic to $S^3 \times \mathbb{R}^3$.

Hint: Polar decomposition for both parts, orthogonal/unitary and positive definite symmetric/hermitian. Then $SO_2(\mathbb{R}) \cong S^1$ and $SU_2(\mathbb{C}) \cong S^3$.

Referenced in: Spring 2024-1.

(a) For $A \in SL_2(\mathbb{R})$, we can find the polar decomposition $A = OP$ for O an orthogonal matrix and P a positive semidefinite matrix. Since $\det(A) = 1$, we have $\det(O)\det(P) = 1$ implying that $\det(O) = \det(P) = 1$ as $\det(O) = \pm 1$ for orthogonal matrices and $\det(P) \geqslant 0$ for positive semidefinite matrices. Thus $O \in SO_2(\mathbb{R})$ and P is positive definite. Now, recall that $SO_2(\mathbb{R}) \cong S^1$ since a special orthogonal 2×2 matrix is just rotation by some $\theta \in [0, 2\pi)$ which we can map diffeomorphically to S^1 via $\theta \mapsto e^{i\theta}$.

On the other hand, P is a positive definite matrix with determinant 1. So

$$P = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$$

with $ad - b^2 = 1$. Hence P is determined uniquely by $a > 0$ and $b \in \mathbb{R}$ since $d = \frac{1+b^2}{a}$ is forced by the determinant condition. Conversely, any choice of $a > 0, b \in \mathbb{R}$ yields a positive definite matrix with determinant 1 of the above form (by Sylvester's criterion). I.e., the positive definite matrices of determinant 1 are diffeomorphic to $(0, \infty) \times \mathbb{R} \cong \mathbb{R}^2$. Finally, it is easy to see that any choice of special orthogonal 2×2

matrix and positive definite 2×2 matrix with determinant 1 gives a matrix with determinant 1 via $A = OP$ so we have the desired diffeomorphism $SL_2(\mathbb{R}) \cong S^1 \times \mathbb{R}^2$.

(b) Now, polar decomposition gives $A = UP$ for U unitary and P positive semidefinite and Hermitian. Similarly, we have $\det(U) = \det(P) = 1$ so $U \in SU_2(\mathbb{C})$ and P is positive definite, Hermitian, and has determinant 1. In this case, we recall that

$$SU_2(\mathbb{C}) \cong S^3 \text{ via } \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \leftrightarrow (a,b) \in S^3 \subset \mathbb{C}^2.$$

On the other hand, P must be of the form

$$P = \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix}$$

for $a, b, c \in \mathbb{C}$ with $ac - |b|^2 = 1$. By Sylvester's criterion, such a matrix is positive definite if and only if $\mathbb{R} \ni a > 0, ac - |b|^2 > 0$ and the determinant condition is satisfied if and only if $c = \frac{1+|b|^2}{a}$. I.e., the positive definite Hermitian matrices of determinant 1 are in bijection with ordered pairs $(a,b) \in (0,\infty) \times \mathbb{C} \cong \mathbb{R}^3$ so we have the desired diffeomorphism $SL_2(\mathbb{C}) \cong S^3 \times \mathbb{R}^3$.

Fall 2012- 2. For $n \geqslant 1$, construct an everywhere non-vanishing smooth vector field on the odd-dimensional real projective space $\mathbb{RP}^{2n-1}$.

Hint: Standard nowhere zero vector field on S^n, $(x_1, x_2, \ldots, x_n, x_{n+1}) \mapsto (-x_2, x_1, \ldots, -x_{n+1}, x_n)$, factors through a vector field on $\mathbb{RP}^n$.

This is the second part of Fall 2019-3.

Fall 2012-3. Let $M^m \subset \mathbb{R}^n$ be a smooth submanifold of dimension $m < n-2$. Show that its complement $\mathbb{R}^n - M$ is connected and simply connected.

Hint: Homotope a path in $\mathbb{R}^n$ to one that is transversal to M and show this must not intersect M. Use extension theorem on a homotopy from $[0,1] \times [0,1] \to M$ with $C = \{0,1\} \times [0,1] \cup [0,1] \times \{0,1\}$ a closed subset with the image of $H|_C$ transverse to M.

This is exactly Fall 2015-6.

Fall 2012-4. (a) Show that for any $n \geqslant 1$ and $k \in \mathbb{Z}$, there exists a continuous map $f : S^n \to S^n$ of degree k.

(b) Let X be a compact, oriented n-dimensional manifold. Show that for any $k \in \mathbb{Z}$, there exists a continuous map $f : X \to S^n$ of degree k.

Hint: Construct $f_k : S^n \to S^n$ of degree k using $z \mapsto z^k$ in S^1 and suspending. $B \subset M$ homeomorphic to $\mathbb{R}^n$, show $q : M \to M/(M - B) \cong S^n$ has degree 1 using a commutative square. Use Whitney's approximation theorem to get smooth map.

This is exactly Fall 2022-3.

Fall 2012-5. Assume that $\Delta = \{X_1, \ldots, X_k\}$ is a k-dimensional distribution spanned by vector fields on an open set $\Omega \subset M^n$ in an n-dimensional manifold. For each open subset $V \subset \Omega$ define

$$Z_V = \{u \in C^\infty(V) \mid X_1 u = 0, \ldots, X_k u = 0\}$$

Show that the following two statements are equivalent:

(a) The distribution Δ is integrable.

(b) For each $x \in \Omega$ there exists an open neighborhood $x \in V \subset \Omega$ and $n-k$ functions $u_1, \ldots, u_{n-k} \in Z_V$ such that the differentials $du_1, \ldots, du_{n-k}$ are linearly independent at each point in V.

Hint: Find chart for which $N = \{x^{k+1}(q) = a^{k+1}, \ldots, x^n(q) = a^n\}$ and the coordinate functions are the desired functions. Check that if η is a 1-form which annihilates Δ, then $d\eta$ also annihilates Δ, using $du_1, \ldots, du_{n-k}$ span Z_V.

(a) $\implies$ (b). For each $p \in \Omega$, there exists an integral submanifold N for Δ at p. We can find a slice chart (V, x) for N centered at p, i.e., $x : V \to (-\epsilon, \epsilon)^n$ and $x(p) = 0$, such that

$$N \cap V = \{q \in V \mid x^{k+1}(q) = \cdots = x^n(q) = 0\}.$$

For $q \in N \cap V$, note that

$$T_qN = \text{Span}\{\frac{\partial}{\partial x^i}\Big|_q \mid i = 1, \ldots, k\} = \Delta_q = \text{Span}\{(X_1)_q, \ldots, (X_k)_q\}.$$

Then, since $\frac{\partial}{\partial x^i}x^j = 0$ for $i \neq j$, we have that $x^{k+1}, \ldots, x^n$ are killed by all vectors in $T_qN = \Delta_q$. In particular, for each $q \in N \cap V$, $(X_i)_q(x^j) = 0$ for $1 \leqslant i \leqslant k$ and $k+1 \leqslant j \leqslant n$. Hence $x^j \in Z_V$ for all $k+1 \leqslant j \leqslant n$. Finally, note that the differentials $dx^{k+1}, \ldots, dx^n$ are clearly linearly independent at each point in V since the x^i are coordinate functions so letting $u_i = x^{k+i}$ gives the desired $n-k$ functions on Z_V.

(b) $\implies$ (a). By Frobenius' theorem, Δ is integrable if and only if Δ is involutive which occurs if and only if the annihilator ideal

$$I(\Delta) = \{\eta \in \Omega^j(M) \mid 1 \leqslant j \leqslant n \text{ and } \eta(Y_1, \ldots, Y_j) = 0 \text{ for all } Y_1, \ldots, Y_j \in \Delta\}$$

is closed under the exterior derivative. It suffices to check that if η is a smooth closed 1-form which annihilates Δ (locally on V), then $d\eta$ also annihilates Δ. By the linear independence of $du_1, \ldots, du_{n-k}$, we can tell that they generate the degree 1 part of $I(\Delta)$ at each point, so if $\eta \in I(\Delta)$, then $\eta = \sum_{i=1}^{n-k} f_i du_i$ for some smooth functions f_i.

Then, $d\eta = \sum_{i=1}^{n-k} df_i \wedge du_i$. For $X, Y \in \Delta$, we have $(df_i \wedge du_i)(X, Y) = df_i(X)du_i(Y) - df_i(Y)du_i(X) = 0 - 0 = 0$ since du_i annihilates Δ. Thus, we see that $d\eta(X, Y) = 0$ and we conclude if η is a 1-form annihilating Δ, so too is $d\eta$ as desired.

Fall 2012-6. On $\mathbb{R}^n - \{0\}$ define the $(n-1)$-forms

$$\sigma = \sum_{i=1}^{n} (-1)^{i-1} x^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$$

$$\omega = \frac{1}{|x|^n} \sum_{i=1}^{n} (-1)^{i-1} x^i dx^1 \wedge \cdots \widehat{dx^i} \wedge \cdots \wedge dx^n$$

(a) Show that $\omega = r^* \circ i^*(\sigma)$, where $i : S^{n-1} \to \mathbb{R}^n - \{0\}$ is the natural inclusion of the unit sphere and $r(x) = \frac{x}{|x|} : \mathbb{R}^n - \{0\} \to S^{n-1}$ the natural retraction.
(b) Show that σ is not a closed form.
(c) Show that ω is a closed form that is not exact.

Hint: (a) $\sigma_p(X_1, \ldots, X_{n-1}) = \det\begin{pmatrix} p & (X_1)_p & \cdots & (X_{n-1})_p \end{pmatrix}$. Check pointwise and do it for bases of the components $T_pS_p^{n-1}$ and $N_pS_p^{n-1}$. (b) Just compute. (c) Use part (a) to show closed, not exact since integrates to $n \cdot \text{vol}(B) > 0$.

(a) First, we claim that

$$\sigma_p(X_1, \ldots, X_{n-1}) = \det\begin{pmatrix} p & (X_1)_p & \cdots & (X_{n-1})_p \end{pmatrix}.$$

For this, it suffices to plug in the basis vectors $\frac{\partial}{\partial x^1}, \ldots, \widehat{\frac{\partial}{\partial x^i}}, \ldots, \frac{\partial}{\partial x^n}$. The right hand side then becomes

$$\det\begin{pmatrix} p & e_1 & \cdots & \hat{e_i} & \cdots & e_n \end{pmatrix} = (-1)^{i-1} \det\begin{pmatrix} e_1 & \cdots & e_{i-1} & p & e_{i+1} & \cdots & e_n \end{pmatrix} = (-1)^{i-1} x^i(p),$$

which agrees with the coefficients of σ. To show $\omega = r^*i^*\sigma$, it suffices to check pointwise and for a fixed $p \in \mathbb{R}^n - \{0\}$, it suffices to check by plugging in basis vectors. We choose a basis of $T_p\mathbb{R}^n = T_pS_p^{n-1} \oplus N_pS_p^{n-1}$ by selecting a basis for each component, where S_p^{n-1} is the unique $(n-1)$-sphere containing p. For $N_pS_p^{n-1}$, we simply take the basis $\{p\}$.

First, we check that if any of the $(X_i)_p$'s are in $N_pS_p^{n-1}$, then $(r^*i^*\sigma)_p(X_1, \ldots, X_{n-1})$ and $\omega_p(X_1, \ldots, X_{n-1})$ are both 0. Write $(X_i)_p = \lambda p$. Then by the determinant form of σ above, we see immediately that $\sigma_p(X_1, \ldots, X_{n-1}) = 0$. So $\omega_p(X_1, \ldots, X_{n-1}) = \frac{1}{|x|^n}\sigma_p(X_1, \ldots, X_{n-1}) = 0$ as well. On the other hand,

$$(r^*i^*\sigma)_p(X_1, \ldots, X_{n-1}) = \sigma_{\frac{p}{|p|}}(d(i \circ r)_pX_1, \ldots, d(i \circ r)_pX_{n-1}).$$

Taking $\gamma(t) = t\lambda p$, we see $\gamma'(1) = \lambda p$. Meanwhile $r \circ \gamma = \frac{p}{|p|}$ is constant, so that $(dr)_p(\lambda p) = 0$. Then $d(i \circ r)_p(X_i)_p = (di)_{p/|p|}dr_p(X_i)_p = 0$ for $(X_i)_p = \lambda p$. From this, we see

$$\sigma_{\frac{p}{|p|}}(d(i \circ r)_pX_1, \ldots, d(i \circ r)_pX_{n-1}) = 0.$$

Hence $\omega_p(X_1, \ldots, X_{n-1}) = (r^*i^*\sigma)_p(X_1, \ldots, X_{n-1}) = 0$ whenever any $(X_i)_p \in N_pS_p^{n-1}$. By the above remarks, it only remains to check on the basis vectors in $T_pS_p^{n-1}$. However, if $X_p \in T_pS^{n-1}$, we may write it as $(dj_p)_pY_p$, where $j_p : S_p^{n-1} \to \mathbb{R}^n - \{0\}$ is inclusion, for some vector field Y on S_p^{n-1}. Thus it remains to check

$$(r^*i^*\sigma)_p((dj_p)_p(Y_1)_p, \ldots, (dj_p)_p(Y_{n-1})_p) = \omega_p((dj_p)_p(Y_1)_p, \ldots, (dj_p)_p(Y_{n-1})_p).$$

However, for this, it simply suffices to check that $j_p^*(r^*i^*\sigma) = j_p^*(\omega)$. Note that $i \circ r \circ j_p = x/|p|$ is just multiplication by $1/|p|$, so that $(i \circ r \circ j_p)^*\sigma$ is $\frac{1}{|p|^n}\sigma|_{S_p^{n-1}}$, where we gain a $1/|p|$ factor from each x_i and dx^j term. Meanwhile,

$$j_p^*\omega = j_p^*(\frac{1}{|x|^n}\sigma) = \frac{1}{|x|^n \circ j_p}j_p^*\sigma = \frac{1}{|p|^n}j_p^*\sigma = \frac{1}{|p|^n}\sigma|_{S_p^{n-1}},$$

since $|x|^n \circ j_p = |p|^n$ is constant.

(b) We can just compute

$$d\sigma = \sum_{i=1}^n (-1)^{i-1}dx^i \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n = \sum_{i=1}^n dx^1 \wedge \cdots \wedge dx^i \wedge \cdots \wedge dx^n = n(dx^1 \wedge \cdots \wedge dx^n),$$

which is nonzero so σ is not closed.

(c) We have $\omega = r^*i^*\sigma$ so $d\omega = d(r^*i^*\sigma) = r^*i^*(d\sigma) = 0$ since $d\sigma$ is an n-form so $i^*d\sigma$ is an n-form on S^{n-1} so must be zero. So ω is closed. On the other hand, $i^*\omega = i^*r^*i^*\sigma = (r \circ i)^*i^*\sigma = i^*\sigma$ since $r \circ i = \text{id}$. But from the expression for σ, we can find a form $\widehat{\sigma}$ on $\mathbb{R}^n$ such that $\sigma = j^*\widehat{\sigma}$ for $j : \mathbb{R}^n - \{0\} \hookrightarrow \mathbb{R}^n$. Then, $d\widehat{\sigma} = n(dx^1 \wedge \cdots \wedge dx^n)$ by the same calculation as in part (b) so Stokes' theorem gives us

$$\int_{S^{n-1}} i^*\omega = \int_{S^{n-1}} i^*\sigma = \int_{S^{n-1}} i^*j^*\widehat{\sigma} = \int_B d\widehat{\sigma} = \int_B n(dx^1 \wedge \cdots \wedge dx^n) = n \cdot \text{vol}(B) \neq 0,$$

where $B = \{x \in \mathbb{R}^n \mid |x| \leqslant 1\}$ is the closed unit n-ball. But if ω were exact, we would have $\omega = d\theta$ so that $i^*\omega = i^*d\theta = d(i^*\theta)$ is exact and hence integrates to 0, a contradiction so ω is not exact.

Fall 2012-7. Let $n \geqslant 0$ be an integer. Let M be a compact, orientable, smooth manifold of dimension $4n+2$. Show that $\dim H^{2n+1}(M;\mathbb{R})$ is even.

Hint: $H^{2n+1}(M) \times H^{2n+1}(M) \to \mathbb{R}$ by $(\omega, \eta) \mapsto \int_M \omega \wedge \eta$. Show this is bilinear, anti-symmetric, and non-degenerate. Corresponds to an invertible $k \times k$ ($k = \dim(H^{2n+1}(M))$) matrix with $A = -A^T$ but if k is odd, we get $\det(A) = -\det(A)$, a contradiction.

This is exactly Fall 2021-5.

Fall 2012-8. Show that there is no compact three-dimensional manifold M whose boundary is the real projective space $\mathbb{RP}^2$.

Hint: Immediate from $\chi(M) = \frac{1}{2}\chi(\partial M)$ and $\chi(\mathbb{RP}^2) = 1$. Glue two copies of M together along the boundary to make N. N is closed, odd-dimensional so has $\chi(N) = 0$ by Poincaré duality (use $\mathbb{Z}/2\mathbb{Z}$ coefficients) and then use Mayer-Vietoris to get equation.

By Spring 2022-9, we know $\chi(M) = \frac{1}{2}\chi(\partial M)$. But $\chi(\mathbb{RP}^2) = 1$ and the Euler characteristic of a manifold is an integer, so we get a contradiction if $\partial M = \mathbb{RP}^2$.

Fall 2012-9. Consider the coordinate axes in $\mathbb{R}^n$:

$$L_i = \{(x_1, \ldots, x_n) \mid x_j = 0 \text{ for all } j \neq i\}$$

Calculate the homology groups of the complement $\mathbb{R}^n - (L_1 \cup \cdots \cup L_n)$.

Hint: $H_*(X) = \mathbb{Z}_{(0)} \oplus \mathbb{Z}^{2n-1}_{(n-2)}$. X is homotopy equivalent to the wedge sum of $2n-1$ copies of S^{n-2}.

Note first that $X = \mathbb{R}^n - (L_1 \cup \cdots \cup L_n)$ deformation retracts onto $S^{n-1} - J$ where J is a collection of $2n$ points. Then, we know that $S^{n-1} - \{p\}$ is homotopy equivalent to $\mathbb{R}^{n-1}$ for $p \in S^{n-1}$ a point so $X \cong \mathbb{R}^{n-1} - K$ for K a collection of $2n-1$ points. From here, each neighborhood of a removed point is homotopy equivalent to S^{n-2}. So with a proper deformation, we can isolate these neighborhoods and retract X onto the wedge sum of $2n-1$ copies of S^{n-2}. Hence, we get

$$H_k(X) = H_k\left(\bigvee_{i=1}^{2n-1} S^{n-2}\right) = \begin{cases} \mathbb{Z} & k = 0, \\ \mathbb{Z}^{2n-1} & k = n-2, \\ 0 & \text{otherwise.} \end{cases}$$

We have degenerate cases for $n = 1$ and 2. Namely for $n = 2$, we get $H_k(X) = \mathbb{Z}^4$ for $k = 0$ and 0 otherwise, while for $n = 0$, $X = \varnothing$ so we ignore this case.

Fall 2012-10. (a) Let X be a finite CW complex. Explain how the homology groups of X are related to the homology groups of $X \times S^1$.
(b) For each integer $n \geqslant 0$, give an example of a compact smooth manifold of dimension $2n+1$ such that $H_i(X) = \mathbb{Z}$ for all $i = 0, \ldots, 2n+1$.

Hint: $H_k(X \times S^1) = H_k(X) \oplus H_{k-1}(X)$ by using product CW structure. $\mathbb{CP}^n \times S^1$.

(a) For each $0 \leqslant k \leqslant N$, let X have n_k k-cells, namely $e_1^k, \ldots, e_{n_k}^k$. Consider the CW structure on S^1 given by one 0-cell v and one 1-cell $e = [0,1]$ which is attached to v by the endpoints. Then, the product $X \times S^1$ has cells that are the products of the cells of X and the cells of S^1.

Namely, we have $(n_k + n_{k-1})$ k-cells $e_i^k \times v$ and $e_j^{k-1} \times e$ for $i = 1, \ldots, n_k$ and $j = 1, \ldots, n_{k-1}$. We then have boundaries as follows

$$\partial(e_i^k \times v) = \partial e_i^k \times v + (-1)^k e_i^k \times \partial v = \partial e_i^k \times v,$$
$$\partial(e_j^{k-1} \times e) = \partial e_j^{k-1} \times e + (-1)^{k-1} e_j^{k-1} \times \partial e = \partial e_j^{k-1} \times e,$$

since $\partial v = 0$ and $\partial e = v - v = 0$. So, given the following chain complex for X:

$$0 \to C_n \to \cdots \to C_i \xrightarrow{\partial_i} C_{i-1} \to \cdots \to C_0 \to 0,$$

we have the complex for $X \times S^1$:

$$\cdots \to C_i \oplus C_{i-1} \xrightarrow{(\partial_i, \partial_{i-1})} C_{i-1} \oplus C_{i-2} \to \cdots .$$

Clearly $\ker(\partial_i, \partial_{i-1}) = \ker(\partial_i) \oplus \ker(\partial_{i-1})$ and $\operatorname{im}(\partial_{i+1}, \partial_i) = \operatorname{im}(\partial_{i+1}) \oplus \operatorname{im}(\partial_i)$. So, taking the homologies of these chain complexes, we have

$$H_i(X \times S^1) = H_i(X) \oplus H_{i-1}(X),$$

where we set $H_{-1}(X) = 0$.

(b) First, we know that the homology $\mathbb{CP}^n$ is $\mathbb{Z}$ in all even degrees and 0 in all odd degrees up to $2n$. Thus $\mathbb{CP}^n \times S^1$ has homology $\mathbb{Z}$ in all degrees from 0 to $2n+1$ (inclusive) by part (a).

Spring 2012

Spring 2012-1. Explain in detail from the viewpoint of transversality theory, why the sum of the indices of a vector field with isolated zeroes on a compact orientable manifold M is independent of what vector field we choose.

Hint: Vector field X corresponds to global (by compactness) flow ϕ_t that is a diffeomorphism for small t. This is homotopic to $\phi_0 = \mathrm{id}_M$, independent of X.

This is precisely the Poincaré-Hopf index theorem. Let X be a vector field with isolated zeroes. By compactness, X has only finitely many zeroes, and we can find a global flow ϕ_t corresponding to X (i.e., $\frac{d}{dt}\mid_{t=0} \phi_t = X$.) For small enough t, ϕ_t is a diffeomorphism with fixed points precisely the zeroes of X which are isolated. In particular, ϕ_t is a Lefschetz map. Then, we have

$$\sum_{p \mid X_p = 0} \operatorname{ind}_p(X) = \sum_{p \mid \phi_t(p) = p} L_p(\phi_t) = \Lambda_{\phi_t}.$$

Namely, the Lefschetz number of ϕ_t is the sum of the local Lefschetz numbers, which are identically the indices of the zeroes of X. Now, note that $\phi_0 = \mathrm{id}_M$ so ϕ_t and id_M are homotopic via the backwards flow ϕ_{-t}. Since the Lefschetz number of a map is homotopy invariant, we have $\Lambda_{\phi_t} = \Lambda_{\mathrm{id}_M}$ which is independent of the vector field X that we began with.

Spring 2012-2. Call the index sum in problem 1 the Euler characteristic $\chi(M)$. Explain why the Euler characteristic of a genus g surface (2-sphere with g handles attached) is $2 - 2g$. [Do this explicitly: do NOT appeal to the theorem that the Euler characteristic in the vector field sense indicated is computable from homological information. That comes next!]

Hint: Dunking the donut flow. Index of source and sink is $+1$ while of a saddle is -1 because of eigenvalues being greater/smaller than one.

We construct a vector field on the surface of genus g as follows: imagine dunking an n-holed donut in some kind of liquid and taking the vector field induced by the flow of the liquid after you pull it out. This will have a source at the top, a sink at the bottom, and $2g$ saddles, one at the top and bottom of each of the g holes. Now, we compute the indices of these zeroes.

Both the source and sink have index 1. This is because at the source, $d\phi_t$ has eigenvalues both greater than 1 so $d\phi_t - I$ has both positive eigenvalues and is thus orientation preserving, while at the sink $d\phi_t - I$ has both eigenvalues less than 1 so again $d\phi_t - I$ has positive determinant and is orientation preserving. On the other hand, at a saddle, we have one eigenvalue of $d\phi_t$ greater than one and one less than one, meaning the determinant of $d\phi_t - I$ is negative so it is orientation reversing. Thus, the sum over the indices is indeed $2 - 2g$.

Spring 2012-3. Suppose that M is a triangulated compact orientable manifold, i.e., a manifold M represented as a finite simplicial complex.

(a) Show that the alternating sum of the Betti numbers $b_0 - b_1 + b_2 - \dots$ (where b_k = rank of the kth homology group with real coefficients) is equal to the alternating sum

$$\text{(number of vertices)} - \text{(number of faces)} + \text{(number of 2-simplices)} - \dots$$

(b) Show that there is a vector field with the sum of its indices equal to the number described in part (a).
[You do not need to worry about smoothness of the vector field – just describe how to build it. In part (a), the result should follow from some dimension counting.]

Hint: (a) Use $\ker(\partial_i)$ and $\operatorname{im}(\partial_i)$ and dimension of quotient vector space etc. (b) Suffices to show for n-simplex, define inductively so we get a sink at the center of each face which gives a zero of index $(-1)^k$ for each k-simplex.

(a) Denote by c_i the number of i-simplices of M. Let ∂_i denote the map $C_i \to C_{i-1}$ in the chain complex with $\mathbb{R}$ coefficients, where we define $C_{-1} = 0$. So $C_i/\ker(\partial_i) \cong \operatorname{im}(\partial_i)$ as vector spaces, implying that $\dim_{\mathbb{R}} C_i = \dim_{\mathbb{R}} \ker(\partial_i) + \dim_{\mathbb{R}} \operatorname{im}(\partial_i)$. Clearly, $\dim_{\mathbb{R}} C_i = c_i$ by definition of simplicial homology (the rank doesn't change when using $\mathbb{R}$-coefficients instead of $\mathbb{Z}$-coefficients.) So we have

$$\sum_{i=0}^{n} (-1)^i c_i = \sum_{i=0}^{n} (-1)^i (\dim_{\mathbb{R}} \ker(\partial_i) + \dim_{\mathbb{R}} \operatorname{im}(\partial_i)).$$

Again, since the rank doesn't change when using $\mathbb{R}$-coefficients (by the universal coefficient theorem), we have

$$b_i = \dim_{\mathbb{R}} H_i(X; \mathbb{R}) = \dim_{\mathbb{R}} (\ker(\partial_i)/\operatorname{im}(\partial_{i+1})) = \dim_{\mathbb{R}} \ker(\partial_i) - \dim_{\mathbb{R}} \operatorname{im}(\partial_{i+1}).$$

Hence

$$\begin{aligned}
\sum_{i=0}^{n} (-1)^i b_i &= \sum_{i=0}^{n} (-1)^i (\dim_{\mathbb{R}} \ker(\partial_i) - \dim_{\mathbb{R}} \operatorname{im}(\partial_{i+1})) \\
&= \sum_{i=0}^{n} (-1)^i \dim_{\mathbb{R}} \ker(\partial_i) + \sum_{i=0}^{n} (-1)^i \dim_{\mathbb{R}} \operatorname{im}(\partial_i) - \dim_{\mathbb{R}} \operatorname{im}(\partial_0) + (-1)^{n+1} \dim_{\mathbb{R}} \operatorname{im}(\partial_{n+1}) \\
&= \sum_{i=0}^{n} (-1)^i (\dim_{\mathbb{R}} \ker(\partial_i) + \dim_{\mathbb{R}} \operatorname{im}(\partial_i)) - 0 + 0 \\
&= \sum_{i=0}^{n} (-1)^i c_i,
\end{aligned}$$

as desired.

(b) It suffices to exhibit such a vector field for Δ^n, as then we can glue these vector fields as we glue the simplices to get a vector field on M. Define X on $\Delta^n - \partial \Delta^n$ to be a vector field pointing towards the center of the interior of Δ^n. This will make the center an n-dimensional sink, with corresponding index $(-1)^n$ (we may even insist the vector field near the center is just $p = (x_1, \dots, x_n) \mapsto X_p = (-x_1, \dots, -x_n)$).

Now, $\partial \Delta^n$ is the union of $(n-1)$-simplices so we repeat this process inductively, describing what to do on the interior of each k-simplex. Each simplex will thus contribute a fixed point of index $(-1)^k$, so we will get the sum of the indices to be the sum of $(-1)^k$ times the number of k-simplices, as desired.

Spring 2012-4. Suppose V is a smooth (C^∞) vector field on $\mathbb{R}^3$ that is nonzero at $(0,0,0)$. The vector field is said to be gradient-like at $(0,0,0)$ if there is a neighborhood of $(0,0,0)$ and a nowhere zero smooth function $\lambda(x,y,z)$ on that neighborhood such that λV is the gradient of some smooth function in some (possible smaller) neighborhood of $(0,0,0)$.

(a) Write $V = (P,Q,R)$. Show by example that there are functions P,Q,R for which V is not gradient-like in a neighborhood of $(0,0,0)$.
[Suggestion: the orthogonal complement of V taken at each point would have to be an integrable 2-plane field.]

(b) Derive a general differential condition on (P,Q,R) which is necessary and sufficient for V to be gradient-like in a neighborhood of $(0,0,0)$.

Hint: $\omega = -ydx + xdy + dz$ has $\omega \wedge d\omega \neq 0, V = (-y,x,1)$. If and only if $\omega \wedge d\omega = 0$, implies $\ker(\omega)$ integrable so we can find chart such that $V = f\frac{\partial}{\partial z}$.

Referenced in: Spring 2009-4.

(a) Write $\omega = Pdx + Qdy + Rdz$. If $V = (P,Q,R)$ is gradient-like, then $\lambda\omega = df$ for some nonzero function λ and some function f. Then $\omega = \frac{1}{\lambda}df$ and $d\omega = d(\frac{1}{\lambda}) \wedge df$ so that $\omega \wedge d\omega = \lambda df \wedge d(\frac{1}{\lambda}) \wedge df = 0$. In particular, if we take $P = -y, Q = x, R = 1$, then $\omega = -ydx + xdy + dz$ so $d\omega = 2dx \wedge dy$ and $\omega \wedge d\omega = 2dx \wedge dy \wedge dz \neq 0$ so ω is not gradient-like. The corresponding vector field is $V = (-y,x,1)$.

(b) In part (a), we showed that if V is gradient-like, then its dual ω satisfies $\omega \wedge d\omega = 0$. Conversely, if $\omega \wedge d\omega = 0$, then $\ker(\omega)$ is an integrable distribution, meaning that there is a 2-manifold N containing 0 such that V is normal to N. We may then choose a coordinate system on an open set U containing 0 in which $V = f\frac{\partial}{\partial z}$. Since V is nonvanishing, f is nonzero so taking $\lambda = \frac{1}{f}$, we see $\lambda V = \frac{\partial}{\partial z}$ which is the gradient of $g(x,y,z) = (0,0,z)$ so V is gradient-like. I.e., the necessary and sufficient condition for $V = (P,Q,R)$ to be gradient-like is for $\omega = Pdx + Qdy + Rdz$ to satisfy $\omega \wedge d\omega = 0$.

Spring 2012-5. (a) Define carefully the "boundary map" which defines the H_n to H_{n-1} mapping that arises in the long exact sequence arising from a short exact sequence of chain complexes.

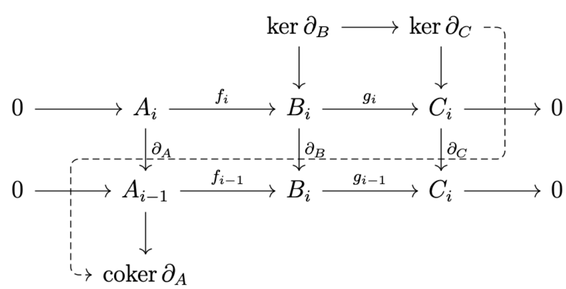
(b) Prove that the kernel of the boundary map is equal to the image of the map into H_n.

Hint: The art of the diagram chase cannot be hinted at. Show part (b) as two directions.

Referenced in: Spring 2008-7.

(a) We described the map in part (b) of Fall 2022-6.

(b) For a general short exact sequence of chain complexes $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$, we have the following set up:



Denote by $\delta : \ker\partial_C \to \mathrm{coker}\partial_A$ the connecting homomorphism. Suppose $\beta \in \ker\partial_B$ maps to $\gamma \in \ker\partial_C$. By the definition in (a), $\delta(\gamma) = \alpha + \mathrm{im}\partial_A \in \mathrm{coker}\partial_A$ where $\alpha \in A_{i-1}$ is the unique element such that $f_{i-1}(\alpha) =$

$\partial_B(\beta)$. But $\beta \in \ker\partial_B$ implies $\partial_B(\beta) = 0$ so $\alpha = 0$ since f_{i-1} is injective and thus $\delta(\gamma) = \text{im}\partial_A = 0 \in \text{coker}\partial_A$. I.e., $\text{im}(\ker\partial_B \to \ker\partial_C) \subset \ker\delta$.

Conversely, suppose $\gamma \in \ker\partial_C$ has $\delta(\gamma) = 0$. By the definition in (a), this means there exists $\alpha \in \text{im}\partial_A$ such that $f_{i-1}(\alpha) = \partial_B(\beta)$ for some $\beta \in B_i$ such that $g_i(\beta) = \gamma$. Suppose $\alpha = \partial_A(a)$ and consider $f_i(a) \in B_i$. By commutativity, $\partial_B \circ f_i(a) = f_{i-1} \circ \partial_A(a) = f_{i-1}(\alpha) = \partial_B(\beta)$ so $\beta - f_i(a) \in \ker\partial_B$. But now, $g_i(\beta - f_i(a)) = g_i(\beta) - g_i(f_i(a)) = \gamma - 0$ by exactness of the top row. I.e., $\ker\delta \subset \text{im}(\ker\partial_B \to \ker\partial_C)$.

Spring 2012-6. Compute the homology of the real projective space $\mathbb{RP}^n$ for each $n > 1$.

Hint: One cell in each dimension with double cover for attaching maps. $H_k(\mathbb{RP}^n) = \mathbb{Z}$ if $k = n$ and n is odd, $\mathbb{Z}/2\mathbb{Z}$ if $k \leqslant n$ and k is odd and 0 otherwise.

The result in Fall 2020-7 generalizes trivially (noting the difference between n even and n odd).

Spring 2012-7. (a) Define complex projective space $\mathbb{CP}^n, (n = 1, 2, 3, \ldots)$.
(b) Show that $\mathbb{CP}^n$ is compact for all n.
(c) Show that $\mathbb{CP}^n$ has a cell decomposition with one cell in each dimension $0, 2, 4, \ldots, 2n$ and no other cells. Include a careful description of the attaching maps.

Hint: Attach by $\phi_n : e^{2n} \to \mathbb{CP}^n, \quad (z_0, \ldots, z_{n-1}) \mapsto [z_0, \ldots, z_{n-1}, t], \quad t = \sqrt{1 - \sum_{i=1}^{n-1} z_i\overline{z_i}}$. Compactness is immediate from CW structure.

(a) and (c) This was done in Spring 2021-5.

(b) $\mathbb{CP}^n$ has a finite CW structure so is certainly compact.

Spring 2012-8. Suppose a compact (real) manifold M has a (finite) cell decomposition with only even dimensional cells. Is M necessarily orientable? Justify your answer.

Hint: Yes. A connected, closed n-manifold is orientable if and only if $H_n(M) = \mathbb{Z}$

Note that a closed, connected n-manifold is orientable if and only if $H_n(M) = \mathbb{Z}$. We suppose that M is path connected. If not, we repeat this argument on each path component to conclude that each path component is orientable. If $n = \dim(M)$, we claim that $H_n(M) = \mathbb{Z}$ so indeed M is orientable. This is because we must have an n-cell as the interior of a k-cell is locally homeomorphic to $\mathbb{R}^k$ and we cannot have an $(n-1)$-cell or an $(n+1)$-cell since n must then be even so $n \pm 1$ are odd. Thus, the map $C_{n+1} \to C_n \to C_{n-1}$ are both 0 and $H_n(M) \cong C_n$ is exactly $\mathbb{Z}$. Further, we know that M has no boundary as if it did, it would be a manifold of dimension $n - 1$ so would require an $(n-1)$-cell which it cannot have.

Spring 2012-9. Suppose that a finite group Γ acts smoothly on a compact manifold M and that the action is free, i.e., $\gamma(x) = x$ for some x in M if and only if $\gamma =$ the identity of the group Γ.

(a) Show that M/Γ is a manifold (i.e., can be made a manifold in a natural way).
(b) Show that $M \to M/\Gamma$ is a covering space.
(c) If the kth de Rham cohomology of M is 0 for some particular $k > 0$, then is the kth de Rham cohomology of M/Γ necessarily 0? Prove your answer.

Hint: $\pi^{-1}(y) = \{g_1x, \ldots, g_nx\}$. Find neighborhoods V_i of g_ix so that $V_i \cong \mathbb{R}^n$ and $g_jV_i = V_k$ where $g_jg_i = g_k$. (c) Finite cover induces injection on de Rham cohomology.

(a) and (b) This is exactly Spring 2017-10.

(c) Yes. By Spring 2019-6, $\pi : M \to M/\Gamma$ induces an injection $\pi^* : H^k_{dR}(M/\Gamma) \to H^k_{dR}(M)$ for any k. So if $H^k_{dR}(M)$ is zero, then $H^k_{dR}(M/\Gamma)$ is also zero.

Spring 2012-10. Let $M = \mathbb{RP}^2 \times \mathbb{RP}^2$ (where $\mathbb{RP}^2$ is real projective 2-space). In a product manifold like that, homology elements can arise by taking in effect the product of a cycle in one factor with a cycle in the other factor. Show that in the case of this particular M, there is an element in the 3-homology with $\mathbb{Z}$ coefficients that does not arise in this way be exhibiting such an element explicitly, e.g. in terms of a cell decomposition.

Hint: Product CW-structure, $\ker(\partial_3) = \{(x, x) \in \mathbb{Z}^2 \mid x \in \mathbb{Z}\}$ and $\operatorname{im}(\partial_4) = \{(x, x) \in \mathbb{Z}^2 \mid x \in 2\mathbb{Z}\}$.

This is exactly Spring 2017-9.

Fall 2011

Fall 2011-1. Let M be an (abstract) compact smooth manifold. Prove that there exists some $n \in \mathbb{Z}^+$ such that M can be smoothly embedded in the Euclidean space $\mathbb{R}^n$.

Hint: Finite cover, collection of bump functions.

$F : M \to \mathbb{R}^{N(m+1)}, x \mapsto (\lambda_1(x)\phi_1(x), \ldots, \lambda_k(x)\phi_k(x), \lambda_1(x), \ldots, \lambda_k(x))$.

Referenced in: Spring 2009-5.

For compact manifolds, an embedding is an injective immersion (properness is immediate). Let $\dim M = m$. Since M is compact, take a finite cover $(U_1, \phi_1), \ldots, (U_n, \phi_N)$ of charts. Let λ_i be a partition of unity subordinate to the cover U_i and define:

$$F : M \to \mathbb{R}^{N(m+1)}, \quad x \mapsto (\lambda_1(x)\phi_1(x), \ldots, \lambda_k(x)\phi_k(x), \lambda_1(x), \ldots, \lambda_k(x)).$$

Suppose that $F(x) = F(y)$. Then $\lambda_i(x) = \lambda_i(y)$ for all i. We know x must be in some U_i as they cover M so some $\lambda_i(x) \neq 0$ so $\lambda_i(x)\phi_i(x) = \lambda_i(x)\phi_i(y)$ implies that $\phi_i(x) = \phi_i(y)$. But the ϕ_i's are all homeomorphisms which are injective so $x = y$ and thus F is injective.

Next, suppose that $v, w \in T_x M$ have $dF_x(v) = dF_x(w)$. Then, $d(\lambda_i)_x(v) = d(\lambda_i)_x(w)$ for all i. Further, we must have

$$d(\lambda_i\phi_i)_x(v) = d(\lambda_i)_x(v)\phi_i(x) + \lambda_i(x)d(\phi_i)_x(v) = d(\lambda_i)_x(w)\phi_i(x) + \lambda_i(x)d(\phi_i)_x(w),$$

implying that

$$\lambda_i(x)d(\phi_i)_x(v) = \lambda_i(x)d(\phi_i)_x(w)$$

for all i. Again, some $\lambda_i(x) \neq 0$ so $d(\phi_i)_x(v) = d(\phi_i)_x(w)$ for some i which implies that $v = w$ since ϕ_i is a diffeomorphism so has bijective derivative locally. Thus F is an immersion since dF_x is injective for all $x \in M$.

Fall 2011-2. Prove that the real projective space $\mathbb{RP}^n$ is a smooth manifold of dimension n.

Hint: Define charts where one coordinate is not zero and send to fraction.

We define charts on $\mathbb{RP}^n$. Let

$$U_i = \{[x_0 : \cdots : x_n] \in \mathbb{RP}^n \mid x_i \neq 0\} \subset \mathbb{RP}^n,$$

for each $0 \leqslant i \leqslant n$. Note that its preimage in S^n is $V_i = \{x \in S^n \mid x_i \neq 0\}$ which is open so U_i is open in the quotient topology. Clearly the U_i's cover $\mathbb{RP}^n$ since $[0 : \cdots : 0] \notin \mathbb{RP}^n$. Define

$$\phi_i : U_i \to \mathbb{R}^n, \quad [x_0 : \cdots : x_n] \mapsto \left(\frac{x_0}{x_i}, \ldots \frac{\widehat{x_i}}{x_i}, \ldots, \frac{x_n}{x_i}\right).$$

This is invariant under scaling so is well-defined. It is also clear that ϕ_i is a homeomorphism. Further, the transition maps are all smooth since $\phi_j \circ \phi_i^{-1}$ is defined by

$$(v_0, \ldots, \widehat{v}_i, \ldots, v_n) \mapsto [v_0 : \cdots : 1 : \cdots : v_n] \mapsto \left(\frac{v_0}{v_j}, \ldots, \frac{\widehat{v_j}}{v_j}, \ldots, \frac{v_n}{v_j}\right),$$

where $v_i := 1$ which is smooth since $v_j \neq 0$ on the intersection $U_i \cap U_j$.

> **Fall 2011-3.** Let M be a compact, simply connected smooth manifold of dimension n. Prove that there is no smooth immersion $f : M \to T^n$, where $T^n = S^1 \times \cdots \times S^1$ is the n-torus.

Hint: Submersion is open map so onto. Local diffeomorphism implies locally even covering and then piece together using compact/connected. Then simply connected covering is universal, a contradiction.

Suppose $f : M \to T^n$ is an immersion. Since $\dim(M) = n = \dim(T^n)$, f is actually a local diffeomorphism. By Spring 2018-1, we know that f is a covering map. Since M is simply connected, we conclude that M is the universal cover of T^n. But we know the universal cover of T^n is $\mathbb{R}^n$ which is not compact, a contradiction so no such immersion exists.

> **Fall 2011-4.** Give a topological proof of the Fundamental Theorem of Algebra: any non-constant single-variable polynomial with complex coefficients has at least one complex root.

Hint: Select $r > 0$ with $\sum_{i=0}^{m-1} \frac{|a_i|}{r^{m-i}} < \frac{1}{2}$. Show $|p(z)t + (1-t)z^m| > 0$ for all $t \in [0,1]$ and z with $|z| = r$. So we can define a homotopy between $(z/r)^m$ and $p(z)/|p(z)|$ from $\{z \in \mathbb{C} \mid |z| = r\}$ to S^1. The latter can be extended while the former has degree m so $m = 0$, a contradiction.

Suppose $p(z) = z^m + \sum_{i=0}^{m-1} a_i z^i, m > 0$ has no roots in $\mathbb{C}$. Notice for $t \in [0,1]$ that

$$p(z)t + (1-t)z^m = z^m + t\left(\sum_{i=0}^{m-1} a_i z^i\right) = z^m\left(1 + t\sum_{i=0}^{m-1} \frac{a_i}{z^{m-i}}\right).$$

Choose $r > 0$ large enough such that $\sum_{i=0}^{m-1} \frac{|a_i|}{r^{m-i}} < \frac{1}{2}$ which is possible since this sum goes to 0 as r goes to infinity. Then, for any $z \in \mathbb{C}$ with $|z| = r$ and for any $t \in [0,1]$, we have

$$\begin{aligned}
|p(z)t + (1-t)z^m| &= \left|z^m\left(1 + t\sum_{i=0}^{m-1} \frac{a_i}{z^{m-i}}\right)\right| \\
&= r^m\left|1 + t\sum_{i=0}^{m-1} \frac{a_i}{z^{m-i}}\right| \\
&\geqslant r^m\left(1 - t\left|\sum_{i=0}^{m-1} \frac{a_i}{z^{m-i}}\right|\right) \\
&\geqslant r^m\left(1 - t\sum_{i=0}^{m-1} \frac{|a_i|}{r^{m-i}}\right) \\
&> r^m(1 - \frac{t}{2}) \geqslant \frac{r^m}{2} > 0.
\end{aligned}$$

In particular, $p(z)t + (1-t)z^m$ is nonzero for $|z| = r$ and all $t \in [0,1]$. So, letting $S_r = \{z \in \mathbb{C} \mid |z| = r\}$, we may define the homotopy

$$H : [0,1] \times S_r \to S^1 \subset \mathbb{C}, \quad H(t,x) = \frac{p(z)t + (1-t)z^m}{|p(z)t + (1-t)z^m|}.$$

By construction, we have $H(0,z) = z^m/r^m = (z/r)^m$ while $H(1,z) = p(z)/|p(z)|$. The map $S_r \to S^1, z \mapsto (z/r)^m$ has degree m and degree is homotopy invariant so $z \mapsto p(z)/|p(z)|$ also must have degree m. But we can extend $z \mapsto p(z)/|p(z)|$ to $D_r^2 = \{z \in \mathbb{C} \mid |z| \leqslant r\}$ since $p(z)$ has no roots so this is always well-defined. So the degree of $z \mapsto p(z)/|p(z)|$ must be 0, implying that $m = 0$, a contradiction so no such $p(z)$ can exist.

Fall 2011-5. Let $f : M \to N$ be a smooth map between two manifolds M and N. Let α be a p-form on N. Show that $d(f^*\alpha) = f^*(d\alpha)$.

Hint: Show induction for function and then for $\alpha = dg \wedge \eta$. Linearity/locality finishes the proof.

Referenced in: Spring 2008-1.

We proceed by induction. First let α be a 0-form, $\alpha = g \in C^\infty(N)$. Then $d(f^*\alpha)(X) = d(g \circ f)(X) = X(g \circ f)$ while $f^*(d\alpha)(X) = (d\alpha)(f_*X) = (f_*X)(g) = X(g \circ f)$ for any vector field X so $d(f^*\alpha) = f^*(d\alpha)$. Next, suppose this holds for $(k-1)$-forms. Let $\alpha = dg \wedge \eta$ for some $(k-1)$-form η and $g \in C^\infty(N)$. Then

$$\begin{aligned} d(f^*\alpha) &= d(f^*(dg \wedge \eta)) = d(f^*dg \wedge f^*\eta) = d(f^*dg) \wedge f^*\eta - f^*dg \wedge d(f^*\eta) \\ &= d(df^*g) \wedge f^*\eta - f^*dg \wedge f^*d\eta = -f^*dg \wedge f^*d\eta = f^*(d(dg \wedge \eta)) = f^*(d\alpha). \end{aligned}$$

Finally, observe that every k-form can locally be written as a sum of terms of the form $gdx_1 \wedge \cdots \wedge dx_k = dx_1 \wedge \eta$ for a $(k-1)$-form η. So it follows by linearity and the fact that it is enough to show this locally that $d(f^*\alpha) = f^*(d\alpha)$ for every form.

Fall 2011-6. (a) What are the de Rham cohomology groups of a smooth manifold?
(b) State de Rham's theorem.

Hint: Cochain complex with exterior derivative as boundary operator.

$H^p_{dR}(M) \cong H^p(M;\mathbb{R}) = \mathrm{Hom}_{\mathbb{R}}(H_p(M),\mathbb{R})$, via $\omega \mapsto \left([c] \mapsto \int_c \omega\right)$.

(a) This is the first part of Spring 2015-7.

(b) For M a smooth manifold and any nonnegative integer p, we have an isomorphism

$$H^p_{dR}(M) \cong H^p(M;\mathbb{R}) = \mathrm{Hom}_{\mathbb{R}}(H_p(M),\mathbb{R}), \text{ via } \omega \mapsto \left([c] \mapsto \int_c \omega\right).$$

Fall 2011-7. Consider the form

$$\omega = (x^2 + x + y)dy \wedge dz$$

on $\mathbb{R}^3$. Let $S^2 = \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$ be the unit sphere, and $i : S^2 \to \mathbb{R}^3$ the inclusion.

(a) Calculate $\int_{S^2} i^*\omega$.
(b) Construct a closed form α on $\mathbb{R}^3$ such that $i^*\alpha = i^*\omega$, or show that such a form α does not exist.

Hint: Stokes': $4\pi/3$. No such form exists since otherwise $\int_{S^2} i^*\omega = 0 \neq 4\pi/3$.

(a) Note that S^2 is the boundary ∂B for $B \subset \mathbb{R}^3$ the unit 3-ball. Thus, using Stokes' theorem

$$\int_{S^2} i^*\omega = \int_B d\omega = \int_B (2x+1)dx \wedge dy \wedge dz = \int_B (2x+1)dV.$$

Now, $\int_B 2xdV = 0$ by the symmetry of the unit ball and $\int_B 1dV = \mathrm{vol}(B) = 4\pi/3$ so $\int_{S^2} i^*\omega = 4\pi/3$.

(b) Suppose α is a closed form on $\mathbb{R}^3$ with $i^*\alpha = i^*\omega$. Then

$$4\pi/3 = \int_{S^2} i^*\omega = \int_{S^2} i^*\alpha = \int_B d\alpha = \int_B 0 = 0,$$

a contradiction so no such α exists.

Fall 2011-8. (a) Let M be a Möbius band. Using homology, show that there is no retraction from M to ∂M.

(b) Let K be a Klein bottle. Show that there exist homotopically nontrivial simple closed curves γ_1 and γ_2 on K such that K retracts to γ_1, but does not retract to γ_2.

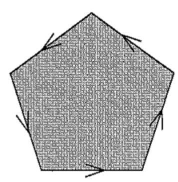
Hint: Use $\mathbb{Z}/2\mathbb{Z}$ coefficients in order to apply Lefschetz duality to simplify the long exact sequence for the pair $(M, \partial M)$. Get $\ker(H_{n-1}(\partial M) \to H_{n-1}(M)) \cong \mathbb{Z}/2\mathbb{Z}$, contradicting the fact that it should be an injection if $r : M \to \partial M$ is a retraction. Look at polygon representation of the Klein bottle, zig-zag shape vs loop around outside.

(a) This follows from Spring 2023-4.

(b) First, note that $K = M \cup_{\partial M} M$ is space obtained by gluing two Möbius bands along their boundaries. Thus, if we take $\gamma_2 = \partial M \subset K$ to be the loop defined by the boundary of either Möbius band, we immediately get, by part (a) that K does not retract onto γ_2 since M does not retract onto $\partial M = \gamma_2$.

On the other hand, we can also treat $K = (S^1 \times [0,1])/\sim$ where $(z,0) \sim (z^{-1},1)$ for any $z \in S^1 \subset \mathbb{C}$. Let $L = \{1\} \times [0,1] \subset S^1 \times [0,1]$ and L' be the image of L under the projection $\pi : S^1 \times [0,1] \to K$. Note that L' is a loop in K since $(1,0) \sim (1,1)$. Now, consider the composition $S^1 \times [0,1] \to L \xrightarrow{\pi} L'$ given by $(z,t) \mapsto (1,t) \mapsto [(1,t)]$. This descends to a well-defined map $f : K \to L'$ since $(z,0) \mapsto [(1,0)] = [(1,1)] \leftarrow\!\shortmid (z,1)$ for any $z \in S^1$. By construction $f \circ i = \mathrm{id}_{L'}$ for $i : L' \to K$ so taking γ_1 to be the loop L' shows that K retracts onto γ_1. However, it is clear that γ_1 is homotopically nontrivial.

Fall 2011-9. Let X be the topological space obtained from a pentagon by identifying its edges as in the picture:



Calculate the homology and cohomology groups of X with integer coefficients.

Hint: CW structure has one 0-cell, one 1-cell, and one 2-cell attached via a degree 5 map. $H_*(X) = \mathbb{Z}_{(0)} \oplus \mathbb{Z}/5\mathbb{Z}_{(1)}$. Universal coefficient theorem. $H^*(X) = \mathbb{Z}^{(0)} \oplus \mathbb{Z}/5\mathbb{Z}^{(2)}$.

We have a CW structure on X with one 0-cell v, one 1-cell a, and one 2-cell A attached via a degree 5 map. X is connected so $H_0(X) = \mathbb{Z}$. Since the boundary of A is attached to a by a degree 5 map, we know that $H_1(X) = \mathbb{Z}/5\mathbb{Z}$. Finally, this map is injective so $\ker(\partial_2) = 0$ implying that $H_2(X) = 0$ and all higher homology groups are trivially zero. To summarize,

$$H_k(X) = \begin{cases} \mathbb{Z} & k = 0, \\ \mathbb{Z}/5\mathbb{Z} & k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

For cohomology, the universal coefficient theorem gives

$$\begin{aligned} H^0(X) &= \mathrm{Hom}_{\mathbb{Z}}(H_0(X), \mathbb{Z}) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(H_{-1}(X), \mathbb{Z}) = \mathbb{Z}, \\ H^1(X) &= \mathrm{Hom}_{\mathbb{Z}}(H_1(X), \mathbb{Z}) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(H_0(X), \mathbb{Z}) = 0, \\ H^2(X) &= \mathrm{Hom}_{\mathbb{Z}}(H_2(X), \mathbb{Z}) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(H_1(X), \mathbb{Z}) = \mathbb{Z}/5\mathbb{Z}, \\ H^k(X) &= 0, k > 2. \end{aligned}$$

Fall 2011-10. Let X, Y be topological spaces and $f, g : X \to Y$ two continuous maps. Consider the space Z obtained from the disjoint union $Y \bigsqcup (X \times [0,1])$ by identifying $(x, 0) \sim f(x)$ and $(x, 1) \sim g(x)$ for all $x \in X$. Show that there is a long exact sequence of the form:

$$\cdots \to H_n(X) \to H_n(Y) \to H_n(Z) \to H_{n-1}(X) \to \cdots.$$

Hint: Two long exact sequences for the pairs $(X \times I, X \times \partial I)$ and (Z, Y) where the relatively homology fits in with an isomorphism.

This was done in Fall 2016-10.

Spring 2011

Spring 2011-1. Show that if V is a smooth vector field on a (smooth) manifold of dimension n and if $V(p)$ is nonzero for some point p, then there is a coordinate system defined in a neighborhood of p, say $(x_1, \ldots, x_n)$, such that on a neighborhood of p, $V =$ the x_1 coordinate vector field.

Hint: Use local flow $\phi : I \times U \to U$, define $\psi(a_1, \ldots, a_n) = \phi(a_1, (0, a_2, \ldots, a_n))$. Show $d\psi_0 = \text{id}$ so inverse function theorem gives local inverse $(y_1, \ldots, y_n) = \psi^{-1}(x_1, \ldots, x_n)$ and this chart is what we want $\frac{\partial}{\partial y_1} = V$.

Referenced in: Spring 2008-2.

Since this is a completely local property, it suffices to work in $\mathbb{R}^n$ with $p = 0$. We may also assume that $V_0 = \frac{\partial}{\partial x_1}|_0$ by rotating and rescaling as necessary. We can then write

$$V_p = \sum_{j=1}^{n} f_j(p) \frac{\partial}{\partial x_j}|_p,$$

where $f_j(0) = \delta_{1j}$ since $V_0 = \frac{\partial}{\partial x_1}|_0$. Let ϕ_t be a local flow corresponding to V near the origin. Namely, find a neighborhood U of 0, an interval $I = (-\epsilon, \epsilon)$, and a smooth map $\phi : I \times U \to U$ such that for all $p \in U$, we have

$$\phi_p : I \to U \text{ satisfies } \phi_p(0) = p \text{ and } \frac{\partial}{\partial t}\phi_p(t) = V_{\phi_p(t)}.$$

Then, define

$$\psi : U \to U, \quad \psi(a_1, \ldots, a_n) = \phi(a_1, (0, a_2, \ldots, a_n)) = \phi_{(0,a_2,\ldots,a_n)}(a_1),$$

where we may shrink U such that this is well-defined. Letting $f = (f_1, \ldots, f_n)$ so that $V_p = f(p)$, we notice

$$\frac{\partial}{\partial x_1}\psi(x_1, \ldots, x_n) = \frac{\partial}{\partial x_1}\phi_{(0,x_2,\ldots,x_n)}(x_1) = V_{\phi_{(0,x_2,\ldots,x_n)}(x_1)} = V_{\psi(x_1,\ldots,x_n)} = f(\psi(x_1, \ldots, x_n)).$$

Now, we claim that $d\psi_0 = \text{id}$. To see this, note that the ith column of $d\psi_0$ is $\frac{\partial}{\partial x_i}\phi(x_1, (0, x_2, \ldots, x_n))|_0$. As we just showed, the first column is thus

$$\begin{aligned}\frac{\partial}{\partial x_1}\phi(x_1, (0, x_2, \ldots, x_n))|_0 &= \frac{\partial}{\partial x_1}\phi_{(0,x_2,\ldots,x_n)}(x_1)|_0 = f(\psi(x_1, \ldots, x_n))|_0 = f(\psi(0)) \\ &= f(\phi(0,0)) = f(0) = (1, 0 \ldots, 0).\end{aligned}$$

While for $2 \leqslant i \leqslant n$, we note that $x_1 = 0$ is fixed so we have

$$\frac{\partial}{\partial x_i}\phi(x_1, (0, x_2, \ldots, x_n))|_0 = \frac{\partial}{\partial x_i}\phi(0, (0, x_2, \ldots, x_n))|_0 = \frac{\partial}{\partial x_i}\phi_{(0,x_2,\ldots,x_n)}(0)|_0 = \frac{\partial}{\partial x_i}(0, x_2, \ldots, x_n)|_0,$$

which is clearly the vector $(0, \dots, 0, 1, 0, \dots, 0)$ with a 1 in the ith position, showing the claim. Thus, by the inverse function theorem, ψ is invertible in some neighborhood of 0. Define $(y_1, \dots, y_n) = \psi^{-1}(x_1, \dots, x_n)$ as a new coordinate system around 0. Then, we have

$$\frac{\partial}{\partial y_1} = \sum_{j=1}^n \frac{\partial x_j}{\partial y_1} \frac{\partial}{\partial x_j}.$$

But using the formula $(x_1, \dots, x_n) = \psi(y_1, \dots, y_n)$, we may compute

$$\begin{aligned}\left(\frac{\partial x_1}{\partial y_1}, \dots, \frac{\partial x_n}{\partial y_1}\right) &= \frac{\partial}{\partial y_1}\psi(y_1, \dots, y_n) = f(\psi(y_1, \dots, y_n)) = f(x_1, \dots, x_n) \\ &= (f_1(x_1, \dots, x_n), \dots, f_n(x_1, \dots, x_n)),\end{aligned}$$

so that

$$\frac{\partial}{\partial y_1} = \sum_{j=1}^n f_j(x_1, \dots, x_n) \frac{\partial}{\partial x_j} = V.$$

Spring 2011-2. (a) Demonstrate the formula $L_x = di_X + i_x d$, where L is the Lie derivative and i is the interior product.

(b) Use this formula to show that a vector field X on $\mathbb{R}^3$ has a flow (locally) that preserves volume if and only if the divergence of X is everywhere 0.
[Here divergence is the classical operator that takes a vector field with components f, g, h to the function $f_x + g_y + h_z$, in the usual partial derivative notation $f_x = \frac{\partial f}{\partial x}$, etc.]

Hint: Work locally, d commutes with pullbacks so $L_X(d\omega) = d(L_X\omega)$ and use induction, product rule and product rule for contraction. (b) ϕ preserves volume if and only if $\phi_t^*\omega = \omega$ for small t where $\omega = dx \wedge dy \wedge dz$ is the standard volume form on $\mathbb{R}^3$. Show this if and only if $L_X\omega = d\iota_X\omega = 0$.

(a) We work locally. Let ϕ be a flow corresponding to X so that by definition

$$L_X\omega = \lim_{h\to 0} \frac{\phi_h^*(\omega) - \omega}{h}.$$

We know the exterior derivative commutes with pullbacks so $L_X(d\omega) = d(L_x\omega)$. Given a 0-form f, we have

$$L_X f = Xf = \lim_{h\to 0} \frac{f \circ \phi_h - f}{h} = df(X).$$

On the other hand,

$$d(i_X f) + i_X(df) = d(0) + df(X) = df(X),$$

so the formula holds for 0-forms. Now, suppose the formula holds for all $(k-1)$-forms and consider $df \wedge \eta$ where η is some $(k-1)$-form. Then, we have

$$\begin{aligned}L_X(df \wedge \eta) &= L_X(df) \wedge \eta + df \wedge L_X(\eta) = d(L_X(f)) \wedge \eta + df \wedge (i_X d\eta + di_X\eta) \\ &= ddf(X) \wedge \eta + df \wedge i_X d\eta + df \wedge di_X\eta,\end{aligned}$$

and we also have

$$\begin{aligned}di_X(df \wedge \eta) + i_X d(df \wedge \eta) &= d(i_X(df) \wedge \eta - df \wedge (i_X\eta)) - i_X(df \wedge d\eta) \\ &= d(df(X) \wedge \eta - df \wedge (i_X\eta)) - (i_X df) \wedge d\eta + df \wedge (i_X d\eta) \\ &= ddf(X) \wedge \eta + df(X) \wedge d\eta + df \wedge (di_X\eta) - df(X) \wedge d\eta + df \wedge (i_X d\eta) \\ &= ddf(X) \wedge \eta + df \wedge (di_X\eta) + df \wedge (i_X d\eta) = L_X(df \wedge \eta).\end{aligned}$$

So the formula works for k-forms of the form $f df_1 \wedge \cdots \wedge df_k$. Then, by linearity, this holds for all forms by induction.

(b) This is exactly part (b) of Spring 2017-4.

Spring 2011-3. (a) Explain some systematic reason why there is a closed 2-form on $\mathbb{R}^3 - \{(0,0,0)\}$ (Euclidean 3-space with one point removed) that is not exact. You may do this by exhibiting such a form explicitly and checking that it is closed but not exact or your may argue using theorems that such a form must exist.

(b) With φ such a form (as in part (a)), discuss why, for any smooth map $F : S^2 \to S^2$,

$$\frac{\int_{S^2} F^*\varphi}{\int_{S^2} \varphi} = \deg(F).$$

[Note that this includes explaining why the denominator integral cannot be 0.]

Hint: $\mathbb{R}^3 - \{0\} \cong S^2$ which has non-trivial second cohomology. $\int_{S^n} F^*\omega = \int_{F_*S^n} \omega$ since we can view S^n as an n-cycle in the domain S^n.

(a) We know $\mathbb{R}^3 - \{0\}$ deformation retracts onto S^2 so they have the same homology groups. We know $H_0(S^2) = \mathbb{Z}$ so by Poincaré duality, since S^2 is a closed, oriented 2-manifold, $H^2(S^2) = \mathbb{Z}$ which means that $H^2(S^2;\mathbb{R}) = \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}$ and by de Rham's theorem, $H^2_{dR}(\mathbb{R}^3 - \{0\}) = H^2_{dR}(S^2) = \mathbb{R}$ is nontrivial so there are closed forms that are not exact on $\mathbb{R}^3 - \{0\}$.

(b) By Spring 2011-4, φ is not exact so $\int_{S^2} \varphi \neq 0$. Then, the equality is shown in part (a) of Spring 2013-7.

Spring 2011-4. Show without using de Rham's Theorem (but you may use the Poincaré Lemma without proof), that a 2-form φ on the 2-sphere S^2 that has integral 0 is exact, i.e., equal to $d\omega$ for some 1-form ω on S^2.

Hint: Split into $A = S^2 - \{N\}$ and $B = S^2 - \{S\}$ which are homotopy equivalent to $\mathbb{R}^2$ so we can find an $\omega_{A,B}$ on each of these. Glue together using the S^1 version of this result which says since $\omega_A - \omega_B$ is closed, it is exact on the intersection so we can make ω global via a bump function.

Referenced in: Spring 2011-3, Spring 2009-1.

Let $A = S^2 - \{n\}$ and $B = S^2 - \{s\}$ where n and s are the north and south poles of the sphere respectively. Let $i_A : A \to S^2$ and $i_B : B \to S^2$ be the inclusions. Then,

$$\int_B i_B^*(\varphi) = \int_A i_A^*(\varphi) = \int_{S^2} \varphi = 0,$$

since A and S^2 differ only be a set of measure zero and similarly for B. But A and B are homotopy equivalent to $\mathbb{R}^2$ which is contractible. Now, note that $d\varphi = 0$ since φ is a top form so $d(i_A^*(\varphi)) = i_A^*(d\varphi) = 0$ and $d(i_B^*(\varphi)) = 0$ similarly. Thus, by Poincaré's lemma, $i_A^*(\varphi)$ and $i_B^*(\varphi)$ are exact, say $i_A^*(\varphi) = d\eta$ and $i_B^*(\varphi) = d\gamma$. Now, consider the form $\eta - \gamma$ on $A \cap B$ and we see that

$$d(i_{A\cap B}^*(\eta - \gamma)) = i_{A\cap B}^*(i_A^*(\varphi) - i_B^*(\varphi)) = i_{A\cap B}^*(\varphi - \varphi) = 0,$$

since again, d commutes with pullbacks. Note that $A \cap B$ deformation retracts onto S^1. Then, since $i_{A\cap B}^*(\eta - \gamma)$ is closed and integrates to 0 (because ω does over S^2), the S^1 version of this result implies that $\eta - \gamma = df$ for some 1-form df.

So we may choose a smooth bump function such that on a small neighborhood of the south pole, we have $\zeta = \eta + df$ and $\zeta = \gamma$ on a small neighborhood of the north pole. In this case, this gives a valid gluing of ζ to make it a global form which satisfies $d\zeta = \varphi$

Spring 2011-5. Suppose that $V : U \to S^2$ is a smooth map, considered as a vector field of unit vectors, where $U = \mathbb{R}^3$ with a finite number of points $p_1, \dots, p_n$ removed, all of which lie strictly inside the unit sphere S^2.

Explain carefully, from basic facts about critical values and critical points and the like, why the degree of $V|_{S^2} : S^2 \to S^2$ is equal to the sum of the indices of the vector field V at the points $p_1, \dots, p_n$.

Hint: Neighborhood around each point. Outward pointing normal vector points into each of these disks. Extension theorem says $\deg(V|_{\partial W}) = 0$.

The index of a point p_i is the degree of the map $V|_{S_i} : S_i \to S^2$ where $S_i = \partial D_i$ for some closed 3-ball in the interior of S^2 containing p_i but not containing p_j for $j \neq i$. This degree is independent of the choice of D_i. Let $W = D^3 - \bigcup_i D_i$. Giving ∂W an outward pointing normal is like giving the disjoint union of S^2 and each of the $\partial D_i = S_i$ a normal vector which points outward for S^2 and inward for each S_i.

The degree is the signed sum of preimages of a regular value so since ∂W is a disjoint union, we have

$$\deg(V|_{\partial W}) = \deg(V|_{S^2}) - \sum_{i=1}^{n} \deg(V|_{S_i}),$$

where the right term is exactly the sum of the indices. Note that we can extend V to all of W which means that $\deg(V_{\partial W})$ must be 0 by the extension theorem so we get the desired equality.

Spring 2011-6. (a) Explain what a short exact sequence of chain complexes is.

(b) Describe how a short exact sequence of chain complexes gives rise to a long exact sequence in homology. Include how the connecting homomorphism (where the dimension changes) arises. You do not need to prove exactness of the sequence.

Hint: Chain map (commutes with partials) that makes an exact sequence at each degree).

(a) Let $A_\bullet, B_\bullet, C_\bullet$ be three chain complexes. Given chain maps, $f_\bullet : A_\bullet \to B_\bullet$ and $g_\bullet : B_\bullet \to C_\bullet$, we say that

$$0 \to A_\bullet \xrightarrow{f_\bullet} B_\bullet \xrightarrow{g_\bullet} C_\bullet \to 0$$

is a short exact sequence of chain complexes if at every degree $i \in \mathbb{Z}$, the following sequence

$$0 \to A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i \to 0$$

is a short exact sequence of abelian groups. Note that $f_\bullet$ and $g_\bullet$ being chain maps means they commute with the boundary operators as follows: $\partial \circ f_i = f_{i-1} \circ \partial$ for all $i \in \mathbb{Z}$ and similarly for g.

(b) This is done in part (b) of Fall 2022-6.

Spring 2011-7. (a) Define complex projective space $\mathbb{CP}^n, n = 1, 2, 3, \ldots$.

(b) Compute the homology and cohomology of $\mathbb{CP}^n$ with $\mathbb{Z}$ coefficients. (Any method is allowed. Cell complexes are particularly simple to use. Be sure to explain what the attaching maps are if you adopt this approach.)

Hint: Attach by $\phi_n : e^{2n} \to \mathbb{CP}^n$, $(z_0, \ldots, z_{n-1}) \mapsto [z_0, \ldots, z_{n-1}, t]$, $t = \sqrt{1 - \sum_{i=1}^{n-1} z_i \overline{z_i}}$. $H_*(\mathbb{CP}^n) = \mathbb{Z}_{(0)} \oplus \mathbb{Z}_{(2)} \oplus \cdots \oplus \mathbb{Z}_{(2n)}$.

This was done in Spring 2021-5.

Spring 2011-8. (a) Find the $\mathbb{Z}$-coefficient homology of $\mathbb{RP}^2$ by any systematic method.

(b) Explain explicitly (not using the Künneth Theorem) how a nonzero element of the 3-homology with $\mathbb{Z}$-coefficients of $\mathbb{RP}^2 \times \mathbb{RP}^2$ arises.

Hint: $H_*(\mathbb{RP}^2; \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}_{(1)} \oplus \mathbb{Z}_{(0)}$. Product CW-structure, $\ker(\partial_3) = \{(x, x) \in \mathbb{Z}^2 \mid x \in \mathbb{Z}\}$ and $\operatorname{im}(\partial_4) = \{(x, x) \in \mathbb{Z}^2 \mid x \in 2\mathbb{Z}\}$.

This is exactly Spring 2017-9.

Spring 2011-9. (a) State the Lefschetz Fixed Point Theorem.

(b) Show that the Lefschetz number of any map from $\mathbb{CP}^{2n}$ to itself is nonzero and hence that every map from $\mathbb{CP}^{2n}$ to itself has a fixed point (Suggestion: The action of the map on cohomology with $\mathbb{Z}$ coefficients is determined by what happens to the 2^{nd} cohomology since the whole cohomology ring is generated by the 2^{nd} cohomology).

Hint: If $\Lambda_f \neq 0$, then f has a fixed point where $\Lambda_f = I(\Delta, \text{graph}(f))$ is the intersection number.

$H^*(\mathbb{CP}^{2k}; \mathbb{Z}) = \frac{\mathbb{Z}[\alpha]}{\alpha^{2k+1}}$, $|\alpha| = 2$ so $L(F) = \sum_{j=0}^{2k} m^j$ for some $m \in \mathbb{Z}$.

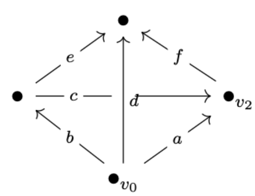
(a) Let $f : X \to Y$ be a smooth map on a compact orientable manifold. If $\Lambda_f \neq 0$, then f has a fixed point, where Λ_f is the Lefschetz number of the map f which is defined to be the intersection number $I(\Delta, \text{graph}(f))$ of two submanifolds of $X \times X$ where $\Delta = \{(x, x) \in X \times X \mid x \in X\}$ and $\text{graph}(f) = \{(x, f(x)) \in X \times X \mid x \in X\}$.

(b) This is exactly Spring 2023-2.

Spring 2011-10. Compute explicitly the simplicial homology with $\mathbb{Z}$ coefficients of the surface of a tetrahedron, thus obtains the homology of the 2-sphere.

Hint: $H_0(X) = H_0(\Delta_3) = \mathbb{Z}$ and $H_1(X) = H_1(\Delta_3) = 0$ for Δ_3 a (contractible) 3-simplex. Compute $H_2(X) = \ker(\partial_2)$ directly by looking at the 2- and 1-simplices of X.

Let Δ_3 be a 3-simplex and let X be a hollow tetrahedron, so X is Δ_3 minus the interior. We know $H_i(X) = \frac{\ker(\partial_i)}{\text{im}(\partial_{i+1})}$ only depends on the i- and $(i+1)$-skeletons. So since X and Δ_3 have the same 0-, 1-, and 2-skeletons, we have $H_0(X) = H_0(\Delta_3) = \mathbb{Z}$ and $H_1(X) = H_1(\Delta_3) = 0$ since the 3-simplex is contractible so has $H_*(\Delta_3) = H_*(*) = \mathbb{Z}$ in degree 0 and 0 otherwise. To compute $H_2(X)$, we consider the following diagram (but imagine relabeling it so it actually agrees with what we say later):



Note that $C_3(X) = 0$ since X has no 3-simplices so $H_2(X) = \ker(\partial_2)$. Now, we explicitly write out the four 2-simplices and the six 1-simplices:

$$\alpha = [v_0, v_1, v_2], \quad \beta = [v_0, v_1, v_3], \quad \gamma = [v_0, v_2, v_3], \quad \delta = [v_1, v_2, v_3]$$

$$a = [v_0, v_1], \quad b = [v_0, v_2], \quad c = [v_0, v_3], \quad d = [v_1, v_2], \quad e = [v_1, v_3], \quad f = [v_2, v_3].$$

And our boundary map acts according to

$$\partial_2(\alpha) = d - b + a, \quad \partial_2(\beta) = e - c + a, \quad \partial_2(\gamma) = f - c + b, \quad \partial_2(\delta) = f - e + d.$$

Let $x = (x_1, x_2, x_3, x_4) \in C_2 = \mathbb{Z}^4$ be in the kernel of ∂_2. By inspection, we see that $x_1 = -x_2$ since the coefficient of a in $\partial_2(x)$, which must be 0, is $x_1 + x_2$. Similarly, we have $x_1 = x_3$ and $x_1 = -x_4$, showing us that $x = (x_1, -x_1, x_1, -x_1)$ but any such x will have $\partial_2(x) = 0$ so $\ker(\partial_2) = \text{Span}((1, 1, -1, -1))$ which is a rank 1 free subgroup of $\mathbb{Z}^4$, implying that $H_2(X) = \mathbb{Z}$. Finally, it is clear that $H_k(X) = 0$ for all $k > 2$ since there are no k-simplices for $k > 2$. To summarize,

$$H_k(X) = \begin{cases} \mathbb{Z} & k = 0, 2, \\ 0 & \text{otherwise.} \end{cases}$$

Fall 2010

Fall 2010-1. Let M be a connected smooth manifold. Show that for any two non-zero tangent vectors v_1 at point x_1 and v_2 at point x_2, there is a diffeomorphism $\phi : M \to M$ such that $\phi(x_1) = x_2$ and $d\phi(v_1) = v_2$.

Hint: Connectedness is required. Map v to w in $\mathbb{R}^n$ ($n \geqslant 2$) using a rotation of the plane spanned by v and w. Need to use flows and 1-parameter subgroup of diffeomorphisms.

This is an easier version of Fall 2020-1.

Fall 2010-2. Let X and Y be submanifolds of $\mathbb{R}^n$. Prove that for almost every $a \in \mathbb{R}^n$, the translate $X + a$ intersects Y transversely.

Hint: Show $F : X \times \mathbb{R}^n \to \mathbb{R}^n, F(x, a) = x + a$ is transverse to Y. Thom's transversality theorem.

This is exactly Spring 2016-2.

Fall 2010-3. Let $M_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$ be the space of $n \times n$ matrices with real coefficients.

(a) Show that
$$SL(n, \mathbb{R}) = \{A \in M_{n\times n}(\mathbb{R}) \mid \det(A) = 1\}$$
is a smooth submanifold of $M_{n\times n}(\mathbb{R})$.

(b) Identify the tangent space to $SL(n, \mathbb{R})$ at the identity matrix I_n.

(c) Show that $SL(n, \mathbb{R})$ has trivial Euler characteristic.

Hint: Preimage theorem with $A \mapsto \det(A)$. $T_I(SL(n, \mathbb{R}) = \ker(d(\det)_I) = \{B \in M_n(\mathbb{R}) \mid \mathrm{Tr}(B) = 0\}$. Homotopy equivalent to $SO(n)$ which is parallelizable so has nowhere vanishing vector field. Also, compact so has $\chi = 0$ by Poincaré-Hopf.

Referenced in: Spring 2009-6.

(a) This is exactly part (a) of Fall 2015-1.

(b) By the regular value theorem, we have

$$T_I SL(n, \mathbb{R}) = \ker(d(\det)_I), \quad d(\det)_I : T_I M_N(\mathbb{R}) = M_N(\mathbb{R}) \to T_1\mathbb{R} = \mathbb{R}.$$

Taking $A = I$ in part (a), we get

$$d(\det)_I(B) = \lim_{h\to 0} \frac{\det(I + hB) - 1}{h} = \mathrm{Tr}(B).$$

This is because, if we let $\lambda_1, \ldots, \lambda_n$ be the generalized eigenvalues of B, then $I + hB$ has generalized eigenvalues $1 + h\lambda_1, \ldots, 1 + h\lambda_n$ so

$$\det(I + hB) = \prod_{i=1}^{n}(1 + h\lambda_i) = 1 + h\mathrm{Tr}(B) + h^2 p(h)$$

for some polynomial $p(h)$ in h. From here, it is clear that the limit is the coefficient of the h term which is exactly the trace of B. Thus, $T_I SL(n, \mathbb{R}) = \{B \in M_n(\mathbb{R}) \mid \mathrm{Tr}(B) = 0\}$.

(c) This is exactly Fall 2020-10

Fall 2010-4. (a) Let $f_i : M \to N, i = 0, 1$, be two smooth maps between smooth manifolds M and N, and $f_i^* : \Omega^*(N) \to \Omega^*(M), i = 0, 1$, be the induced chain maps between the respective de Rham complexes. Define the notion of chain homotopy between f_0^* and f_1^*. Here the co-boundary operators on the de Rham complexes are the exterior derivatives.

(b) Let X be a smooth vector field on a compact smooth manifold M, and let $\phi_t : M \to M$ be the flow generated by X at time t, i.e., the solution of the differential equation $\frac{d\phi_t}{dt}(x) = X(\phi_t(x))$ with initial condition $\phi_0(x) = x$. Find an explicit chain homotopy between the chain maps ϕ_0^* and ϕ_1^*, where $\phi_i^*, i = 0, 1$, are the induced chain maps from $\Omega^*(M)$ to itself.
Hint: Use the formula that for any differential form ω and vector field X, the Lie derivative $L_x\omega = d \circ i_x\omega + i_x \circ d\omega$. Here i_x is the contraction with respect to X.

Hint: $P^{i-1} : \Omega^i(N) \to \Omega^{i-1}(M)$ so that $f_1^i - f_0^i = d \circ P^i + P^{i+1} \circ d$. (b) Cartan's magic formula and fundamental theorem of calculus. Define P by $\omega \mapsto \int_0^1 i_X(\phi_t^*(\omega))dt$.

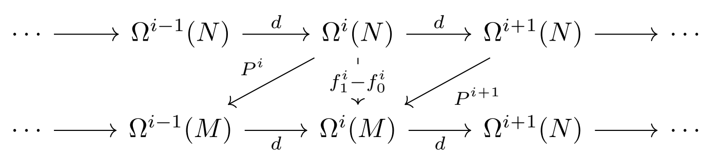
(a) Let f_0^i, f_1^i be the induced map on chains at degree i. Since $\mathrm{Hom}(\Omega^i(N), \Omega^i(M))$ is itself an abelian group, it makes sense to consider $f_1^i - f_0^i$, giving us the following commutative diagram:

$$\begin{CD} \cdots @>>> \Omega^{i-1}(N) @>d>> \Omega^i(N) @>d>> \Omega^{i+1}(N) @>d>> \cdots \\ @. @V{f_1^{i-1}-f_0^{i-1}}VV @V{f_1^i-f_0^i}VV @VV{f_1^{i+1}-f_0^{i+1}}V @. \\ \cdots @>>> \Omega^{i-1}(M) @>>d> \Omega^i(M) @>>d> \Omega^{i+1}(M) @>>d> \cdots \end{CD}$$

We say that f_0^* and f_1^* are chain homotopic if there exists maps $P^i : \Omega^i(N) \to \Omega^{i-1}(M)$ for each i such that

$$f_1^i - f_0^i = d \circ P^i + P^{i+1} \circ d.$$

This can be thought of as the two triangles in the below diagram *adding* to the vertical map (not commuting directly):



(b) Note first that $\phi_0^* = \mathrm{id}$ since $\phi_0 = \mathrm{id}$. We claim the chain homotopy is given by i_X, the interior product by the vector field X. Cartan's magic formula gives us

$$L_X\omega = \lim_{h\to 0} \frac{1}{h}(\phi_h^*\omega - \omega) = i_X(d\omega) + d(i_X\omega).$$

By the fundamental theorem of calculus, and using the above, we have

$$\begin{aligned} (\phi_1^* - \phi_0^*)\omega(p) = \int_0^1 \frac{\partial \phi_t^*(\omega(p))}{\partial t} dt &= \int_0^1 L_X\omega(\phi_t(p))dt \\ &= \int_0^1 di_X(\omega(\phi_t(p))) + i_X(d\omega(\phi_t(p)))dt \\ &= \int_0^1 di_X(\omega(\phi_t(p)))dt + \int_0^1 i_X(d\omega(\phi_t(p)))dt \\ &= d\int_0^1 i_X(\omega(\phi_t(p)))dt + \int_0^1 i_X(d\omega(\phi_t(p)))dt. \end{aligned}$$

So if we define

$$P^i : \Omega^i(M) \to \Omega^{i-1}(M), \quad \omega \mapsto \int_0^1 i_X(\phi_t^*(\omega))dt,$$

we can see that $\phi_1^* - \phi_0^* = d \circ P^i + P^{i+1} \circ d$.

Fall 2010- 5. Let $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + \cdots + dx_{2n-1} \wedge dx_{2n}$ be a 2-form on $\mathbb{R}^{2n}$, where $(x_1, x_2, \ldots, x_{2n})$ are the standard coordinates on $\mathbb{R}^{2n}$. Define an S^1-action on $\mathbb{R}^{2n}$ as follows: for each $t \in S^1$, define $g_t : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ by considering $\mathbb{R}^{2n}$ as the direct sum of n copies of $\mathbb{R}^2$ and rotating each $\mathbb{R}^2$ summand an angle t. Let X be the vector field on $\mathbb{R}^{2n}$ defined by $X(x) = \frac{dg_t(x)}{dt}|_{t=0}$ for any $x \in \mathbb{R}^{2n}$.

(a) Find the Lie derivative $L_X\omega$ and a function f on $\mathbb{R}^{2n}$ such that $df = i_X\omega$.
(b) The S^1-action above induces an action on S^{2n-1}. Let $\mathbb{P}^{n-1}$ be the quotient space of S^{2n-1} by this S^1-action. Show that the quotient space $\mathbb{P}^{n-1}$ has a natural smooth structure and that the tangent space of $\mathbb{P}^{n-1}$ at any point $\underline{x}$ can be identified with the quotient of the tangent space T_xS^{2n-1} by the line spanned by $X(x)$, for any $x \in \underline{x}$. Here $\underline{x}$ is the orbit of x under the S^1-action.
(c) Show that ω descends to a well-defined 2-form on the quotient space $\mathbb{P}^{n-1}$ and that this 2-form so defined is closed.
(d) Is the closed form in (c) exact?
Hint: For (c) and (d), use (a) and (b).

Hint: $f(x_1, \ldots, x_{2n}) = -\frac{1}{2}(x_1^2 + \cdots + x_{2n}^2)$ since we can test $i_X\omega = \omega(X, -)$ on arbitrary Y to get $i_X\omega = -\sum_{i=1}^{2n} x_i dx_i$ so $L_X\omega = 0$. $\mathbb{P}^{n-1}$ is just $\mathbb{CP}^{n-1}$, $X(x)$ is tangent vector in T_xS^{2n-1} that corresponds to rotation in the appropriate complex plane that we are identifying to a point. Descends because $L_X\omega = 0$, closed because original is closed. Not exact because $\omega = d\eta$ with η not descending as it is not invariant under the group action.

(a) We note that
$$X(x_1, \ldots, x_{2n}) = (-x_2, x_1, \ldots, -x_{2n}, x_{2n-1})$$
is the standard nowhere vanishing vector field on S^{2n-1}. Cartan's magic formula gives $L_X\omega = di_X\omega + i_Xd\omega$. First, we consider
$$i_X\omega = \omega(X, -) = \left(\sum_{i=1}^{n} dx_{2i-1} \wedge dx_{2i}\right)(X, -).$$
Locally, at the point $x = (x_1, \ldots, x_{2n})$, we have
$$X = \sum_{i=1}^{n} -x_{2i}\frac{\partial}{\partial x_{2i-1}} + x_{2i-1}\frac{\partial}{\partial x_{2i}},$$
so if $Y = \sum_{j=1}^{2n} y_j \frac{\partial}{\partial y_j}$ is some arbitrary vector field, then
$$dx_{2i-1} \wedge dx_{2i}(X, Y) = \det\begin{pmatrix} -x_{2i} & y_{2i-1} \\ x_{2i-1} & y_{2i}\end{pmatrix} = -(x_{2i-1}y_{2i-1} + x_{2i}y_{2i}).$$
So we can see that
$$i_X\omega = -\sum_{i=1}^{2n} x_i dx_i.$$
Thus, if we set $f(x_1, \ldots, x_{2n}) = -\frac{1}{2}(x_1^2 + \cdots + x_{2n}^2)$, we would have $df = i_X\omega$. On the other hand, $d\omega = 0$ is clear so $L_X\omega = di_X\omega = ddf = 0$ since $d\omega = 0$ is clear.

(b) We claim that $\mathbb{P}^{n-1}$ is just $\mathbb{CP}^{n-1}$ in the following sense. Consider $\mathbb{R}^{2n} = \mathbb{C}^n$ where each pair of coordinates is identified with a copy of $\mathbb{C}$. Then the rotation by S^1 in each plane simply corresponds to complex multiplication by $\lambda \in \mathbb{C}^\times$ for some $|\lambda| = 1$. But this is precisely the identification on $\mathbb{C}^n$ that gives us $\mathbb{CP}^{n-1}$, which we know has a natural smooth structure induced from the quotient $(\mathbb{C}^n - \{0\})/\sim$.

Next, we know that $X(x)$ is the tangent vector in T_xS^{2n-1} that corresponds to the rotation in the appropriate complex plane that we are identifying to a point. Since $T_{\underline{x}}\mathbb{CP}^{n-1}$ has dimension $n-1$, what remains after the quotient by the rotation tangent vector must be the remainder.

(c) We note that ω descends to $\mathbb{CP}^{n-1}$ precisely when ω is invariant with respect to the quotient action. Since this identification is given by the flows of the vector field X, it suffices to show that $L_X\omega = 0$ which was done in part (a). Pulling back the descended form gives the original which is closed so the descended one must be itself closed.

(d) Despite the original form being exact, $\omega = d\eta$, we note that $\eta = \sum_{i=1}^n x_{2i-1}dx_{2i}$ is not invariant under the group action so does not descend to a form on $\mathbb{CP}^{n-1}$

Fall 2010-6. Suppose that $f : S^n \to S^n$ is a smooth map of degree not equal to $(-1)^{n+1}$. Show that f has a fixed point.

Hint: Lefschetz trace fixed point formula.

This is exactly part (a) of Spring 2021-2.

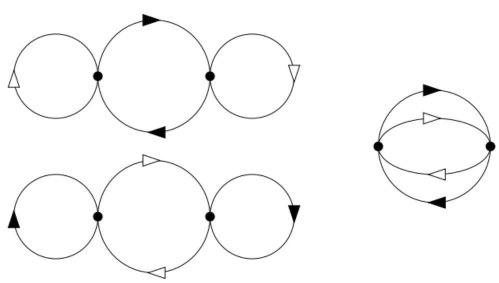
Fall 2010-7. (a) Let G be a finitely presented group. Show that there is a topological space X with fundamental group $\pi_1(X) \cong G$.
(b) Give an example of X in the case $G = \mathbb{Z} * \mathbb{Z}$, the free group on two generators.
(c) How many connected, 2-sheeted covering spaces does the space X from (b) have?

Hint: Fundamental theorem of finitely generated abelian groups. Disjoint union of spaces gives direct sum of homology groups. Attach $(n+1)$-cell to S^n by a degree m map to get $\mathbb{Z}/m\mathbb{Z}$. Three 2-sheeted coverings.

(a) This is exactly Fall 2013-8.

(b) $X = S^1 \vee S^1$.

(c) These correspond to graphs with two vertices, four directed edges, two of which labeled a and two labeled b such that each vertex has two incoming a edges, two outgoing a edges, two incoming b edges, and two outgoing b edges. It is easy to see that there are three such graphs:



Fall 2010-8. Let G be a connected topological group. Show that $\pi_1(G)$ is a commutative group.

Hint: Define a second product which is $\gamma * \alpha : t \mapsto \gamma(t)\alpha(t)$ using the group operation in G. Reparameterize loops to be half the identity to show this is the same as the concatenation operation and also is abelian.

Referenced in: Fall 2008-8.

Define a second product on $\pi_1(G) = \pi_1(G, e)$. For two loops, $\gamma, \alpha : [0,1] \to G$, define

$$[\gamma] * [\alpha] : [0,1] \to G, \quad ([\gamma] * [\alpha])(t) = [\gamma(t)\alpha(t)],$$

where $\gamma(t)\alpha(t) \in G$ is the product in the group G. This is well-defined since $\gamma(0)\alpha(0) = e^2 = e = \gamma(1)\alpha(1)$ so this is still a loop and we can multiply homotopies pointwise to show that this is independent of representatives chosen. We claim that this operation $*$ and the standard concatenation operation $\cdot$ are actually the same on $\pi_1(G, e)$.

Reparameterize γ and α to define

$$\gamma'(t) = \begin{cases} \gamma(2t) & t \in [0, \frac{1}{2}], \\ e & t \in [\frac{1}{2}, 1], \end{cases}, \quad \alpha'(t) = \begin{cases} e & t \in [0, \frac{1}{2}], \\ \alpha(2t-1) & t \in [\frac{1}{2}, 1]. \end{cases}$$

By this construction, we have $[\gamma] = [\gamma']$ and $[\alpha] = [\alpha']$. Moreover, the loop $t \mapsto \gamma'(t)\alpha'(t)$ is exactly the concatenation of γ' and α' so $[\gamma'] * [\alpha'] = [\gamma'] \cdot [\alpha']$, showing that $[\gamma * \alpha] = [\gamma \cdot \alpha]$. Since γ and α were arbitrary, this shows that $*$ and $\cdot$ are indeed the same operation so it suffices to show that $*$ is an abelian operation.

However, this is clear because for any $t \in [0, 1]$, one of $\gamma'(t)$ or $\alpha'(t)$ is the identity by construction so $\gamma'(t)\alpha'(t) = \alpha'(t)\gamma'(t)$ for all $t \in [0, 1]$ as the identity commutes with everything else in G. Thus, $[\gamma] * [\alpha] = [\alpha] * [\gamma]$ and we are done.

Fall 2010-9. Show that if $\mathbb{R}^m$ and $\mathbb{R}^n$ are homeomorphic, then $m = n$.

Hint: Treat $n = 1$ case separately using connectedness on removing a point. Otherwise, remove a point and consider homology of S^{m-1} and S^{n-1}.

First, note that if $m = 1$ and $n \geqslant 2$, then removing a point from $\mathbb{R}^m$ leaves a disconnected space while removing a point from $\mathbb{R}^n$ leaves a connected space so $\mathbb{R}^m$ and $\mathbb{R}^n$ cannot be homeomorphic. Similarly for $n = 1$ so we may assume $m, n \geqslant 2$. Now, suppose $f : \mathbb{R}^m \to \mathbb{R}^n$ is a homeomorphism. Then, removing a point still gives us a homeomorphism $f|_{\mathbb{R}^m - \{p\}} : \mathbb{R}^m - \{p\} \to \mathbb{R}^n - \{f(p)\}$.

But $\mathbb{R}^m - \{p\}$ deformation retracts onto S^{m-1} and $\mathbb{R}^n - \{f(p)\}$ deformation retracts onto S^{n-1} so $\mathbb{R}^m - \{p\} \cong \mathbb{R}^n - \{f(p)\}$ implies that $H_i(S^{m-1}) \cong H_i(S^{n-1})$ for all i. However, we know the homology of a k-sphere is $\mathbb{Z}$ in degree 0 and k, and 0 otherwise. So the only way this can happen is if $m - 1 = n - 1$ so $m = n$.

Fall 2010-10. Let N_g be the nonorientable surface of genus g, that is, the connected sum of g copies of $\mathbb{RP}^2$. Calculate the fundamental group and homology groups of N_g.

Hint: CW structure with 2-cell attached via $e_1^2 \cdots e_g^2$. $H_*(N_g) = \mathbb{Z}_{(0)} \oplus (\mathbb{Z}^{g-1} \oplus \mathbb{Z}/2\mathbb{Z})_{(1)}$.

We give N_g a CW structure with one 0-cell, v, g 1-cells, $e_1, \ldots, e_g$, and one 2-cell, A, which is attached to the 1-skeleton via $e_1^2 \cdots e_g^2$. Hence, we have a cellular chain complex

$$0 \to \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z}^g \xrightarrow{\partial_1} \mathbb{Z} \to 0,$$

where we know that $\partial_1 = 0$ and $\partial_2(A) = 2e_1 + \cdots + 2e_g$ so $\text{im}(\partial_2) = \text{Span}((2, \ldots, 2)) \subset \mathbb{Z}^g$. Hence, we can easily compute

$$H_k(N_g) = \begin{cases} \mathbb{Z} & k = 0, \\ \mathbb{Z}^{g-1} \oplus \mathbb{Z}/2\mathbb{Z} & k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We can also figure out the fundamental group directly from this CW decomposition. Namely,

$$\pi_1(N_g) = \langle e_1, \ldots, e_g \mid e_1^2 \cdots e_g^2 \rangle.$$

Spring 2010

Spring 2010-1. Let M_n be the space of all $n \times n$ matrices with real entries and let S_n be the subset consisting of all symmetric matrices. Consider the map $F : M_n \to S_n$ defined by $F(A) = AA^t - I$, where I is the identity matrix and A^t is the transpose of A.

(a) Show that $0_{n\times n}$ (the $n \times n$ matrix with all entries 0) is a regular value of F.
(b) Deduce that $O(n)$, the set of all $n \times n$ matrices such that $A^{-1} = A^t$ is a submanifold of M_n.
(c) Find the dimension of $O(n)$ and determine the tangent space of $O(n)$ at the identity matrix as a subspace of the tangent space of M_n which is M_n itself.

Hint: $F : \text{Mat}_{n\times n}(\mathbb{R}) \to \text{Sym}_{n\times n}(\mathbb{R})$ by $M \mapsto M^T M$. Then $dF_A(\frac{1}{2}AC) = C$. Preimage theorem. Skew-symmetric matrices.

(a) and (b) This is exactly Fall 2022-2 (though the map is a little different, it yields the same result).

(c) We computed the dimension above. By the regular value theorem,

$$T_I O(n) = \ker(dF_I), \quad dF_I : T_I M_n = M_n \to T_I S_n = S_n.$$

Note that by the above, $dF_I(A) = A^T I + I^T A = A^T + A$ so $T_I O(n) = \{A \in M_n \mid A^T = -A\}$ is the set of skew-symmetric matrices.

Spring 2010-2. Show that $T^2 \times S^n, n \geqslant 1$ is parallelizable, where S^n is the n sphere, $T^2 = S^1 \times S^1$ is the two torus, and a manifold of dimension k is said to be parallelizable if there are k vector fields $V_1, \ldots, V_k$ on it with $V_1(p), \ldots, V_k(p)$ linearly independent for all points p on the manifold.

Hint: Show $S^p \times S^q$ (for p odd) is parallelizable and that the product of parallelizable manifolds is parallelizable. Nowhere vanishing vector field on $S^{2n-1} : (x_1, \ldots, x_{2n}) \mapsto (-x_2, x_1, -x_4, x_3, \ldots, -x_{2n}, x_{2n-1})$. Gymnastics with vector bundles using projection maps and moving two copies of $\mathbb{R}^2$ over to get trivial bundle.

We showed in Fall 2021-2 that $S^p \times S^q$ is parallelizable as long as one of p or q is odd. Thus $S^1 = S^1 \times S^0$ and $S^1 \times S^n$ are both parallelizable. Since $T^2 \times S^n = S^1 \times (S^1 \times S^n)$, the result follows from the fact that the product of parallelizable manifolds is parallelizable. This is because

$$\begin{aligned} T(M \times N) &= \pi_M^* TM \oplus \pi_N^* TN = \pi_M^*(M \times \mathbb{R}^n) \oplus \pi_N^*(N \times \mathbb{R}^m) = M \times N \times \mathbb{R}^n \oplus M \times N \times \mathbb{R}^m \\ &= M \times N \times \mathbb{R}^{m+n}. \end{aligned}$$

Spring 2010-3. Suppose $\pi : M_1 \to M_2$ is a C^∞ map of one connected differentiable manifold to another. And suppose for each $p \in M_1$, the differential $\pi_* : T_p M_1 \to T_{\pi(p)} M_2$ is a vector spae isomorphism.

(a) Show that if M_1 is compact, then π is a covering space projection.
(b) Give an example where M_2 is compact but $\pi : M_1 \to M_2$ is not a covering space (but has the π_* isomorphism property).

Hint: Local diffeomorphism is open map so onto. Local diffeomorphism implies locally even covering and then piece together using compact/connected. $e^{2\pi it} : (-0.1, 1.1) \to S^1$.

This is exactly Spring 2018-1.

Spring 2010-4. Let $\mathcal{F}^k(M)$ denote the differentiable (C^∞) k-forms on a manifold M. Suppose U and V are open subsets of a differentiable manifold.

(a) Explain carefully how the usual exact sequence

$$0 \to \mathcal{F}(U \cup V) \to \mathcal{F}(U) \oplus \mathcal{F}(V) \to \mathcal{F}(U \cap V) \to 0$$

arises.

(b) Write down the "long exact sequence" in de Rham cohomology associated to the short exact sequence in part (a) and describe explicitly how the map

$$H^k_{dR}(U \cap V) \to H^{k+1}_{dR}(U \cup V)$$

arises.

Hint: Suffices to do short exact sequence in each degree as restrictions commute with taking the exterior derivative. Maps are $\omega \mapsto (\omega|_U, \omega|_V)$ and $(\omega, \eta) \mapsto \omega|_{U\cap V} - \eta|_{U\cap V}$. Second part comes from snake lemma, for $\omega \in H^k(U \cap V)$, represent it by $\eta_U - \eta_V$, glue $d\eta_U$ and $d\eta_V$ together on $U \cup V$.

(a) Note that the maps between these chains have degree 0 and we know that the exterior derivative d commutes with pullbacks (restrictions, as we are doing here are a specific case of pullbacks). Thus, it suffices to show that we have a short exact sequence

$$0 \to \mathcal{F}^k(U \cup V) \xrightarrow{f^k} \mathcal{F}^k(U) \oplus \mathcal{F}^k(V) \xrightarrow{g^k} \mathcal{F}^k(U \cap V) \to 0$$

for each k since the commuting with the boundary map part will be automatically satisfied. We define f^k and g^k as follows:

$$f^k(\omega) = (\omega|_U, \omega|_V), \quad g^k(\omega, \eta) = \omega|_{U\cap V} - \eta|_{U\cap V},$$

where $\eta|_W$ is the restriction of the form η to the open set $W \subset M$. Now, we show exactness. Given $\omega \in \mathcal{F}^k(U \cap V)$, we have

$$\omega \mapsto (\omega|_U, \omega|_V) \mapsto \omega|_{U\cap V} - \omega|_{U\cap V} = 0$$

so indeed $\text{im}(f^k) \subset \ker(g^k)$. Conversely, if $(\omega, \eta) \in \ker(g^k)$, then $\omega|_{U\cap V} = \eta|_{U\cap V}$ so we may glue the two forms together to define

$$\theta(x) = \begin{cases} \omega(x) & x \in U, \\ \eta(x) & x \in V - U. \end{cases}$$

Then, $\theta \in \mathcal{F}^k(U \cup V)$ and we clearly have $f^k(\theta) = (\theta|_U, \theta|_V) = (\omega, \eta)$.

(b) The long exact sequence is

$$\cdots \to H^{k-1}(U \cap V) \xrightarrow{\delta} H^k(U \cup V) \xrightarrow{f^k_*} H^k(U) \oplus H^k(V) \xrightarrow{g^k_*} H^k(U \cap V) \to \cdots,$$

where δ is the usual connecting map arising from the snake lemma. In this case, we can explicitly describe it as follows: given $\omega \in H^k(U \cap V)$, we are guaranteed a $\eta_U \in \mathcal{F}^k(U)$ and $\eta_V \in \mathcal{F}^k(V)$ such that ω is given by $\eta_U|_{U\cap V} - \eta_V|_{U\cap V}$. Then, $d\omega = d\eta_U|_{U\cap V} - d\eta_V|_{U\cap V} = 0$ so $d\eta_U|_{U\cap V} = d\eta_V|_{U\cap V}$ meaning we can glue these two together to form a $(k+1)$-form on $U \cup V$.

Spring 2010-5. Explain carefully why the following holds: if $\pi : S^N \to M, N > 1$ is a covering space with M orientable, then every closed k-form on $M, 1 \leqslant k < N$ is exact.

(Suggestion: Recall that the covering transformations in this situation form a group G with $S^N/G \cong M$).

Hint: Finite-sheeted cover induces injection on de Rham cohomology. $U \subset Y$ evenly covered by $f^{-1}(U) = \bigcup_{i=1}^n U_i$ with local inverses $\phi_i : U \to U_i$ to f. Then, define $g(\omega)|_U = \frac{1}{n}\sum_{i=1}^n \phi_i^*\omega$ which is well-defined and satisfies $g \circ f^* = \text{id}$.

Note that S^N is compact so π is a finite-sheeted covering map. Note that

$$H^k_{dR}(S^N) = \begin{cases} \mathbb{R} & k = 0, N, \\ 0 & \text{otherwise.} \end{cases}$$

Then, by Spring 2019-6, we know that $\pi^* : H^k_{dR}(M) \to H^k_{dR}(S^1)$ is injective for all k. In particular, for $1 \leqslant k < N$, this allows us to conclude that $H^k_{dR}(M) = 0$ for $1 \leqslant k < N$, implying that every closed k-form on M is indeed exact.

Spring 2010-6. Calculate the singular homology of $\mathbb{R}^n, n > 1$, with k points removed, $k \geqslant 1$. (Your answer will depend on k and n).

Hint: $H_*(X) = \mathbb{Z}_{(0)} \oplus \mathbb{Z}^k_{(n-1)}$. Wedge sum of k S^{n-1}'s.

We can isolate a neighborhood around each of the removed points, allowing us to deformation retract this space onto the wedge sum of k $(n-1)$-spheres which have homology:

$$H_i\left(\bigvee_{j=1}^k S^{n-1}\right) = \begin{cases} \mathbb{Z} & i = 0, \\ \mathbb{Z}^k & i = n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Spring 2010-7. (a) Explain what is meant by adding a handle to a 2-sphere for a two dimensional orientable surface in general.
(b) Show that a 2-sphere with a positive number of handles attached can not be simply connected.

Hint: Remove two disks and attach a cylinder. Connected sum of tori, induction. $\pi_1(T^2) = \mathbb{Z} \times \mathbb{Z}$ is no trivial.

(a) Remove two 2-disks from the surface and attach a 2-cylinder (with each boundary circle glued to the boundary of one of the removed disks in an orientation-preserving way).

(b) In part (a) of Spring 2018-7, we showed that $\pi_1(M\#N) = \pi_1(M) * \pi_1(N)$. We note that a 2-sphere with $k > 0$ handles attached is the connected sum of $k-1$ tori ($T^2 = S^1 \times S^1$) where the torus is diffeomorphic to a 2-sphere with a single handle attached. Thus, by induction, we have $\pi_1(M_k) = *_{i=1}^{k-1}\pi_1(T^2)$ where M_k is a 2-sphere with k handles attached. But we know $\pi_1(T^2) = \mathbb{Z} \times \mathbb{Z}$ so $\pi_1(M_k) \neq 0$ for all $k > 1$, showing that M_k is not simply connected.

Spring 2010-8. (a) Define the degree $\deg f$ of a C^∞ map $f : S^2 \to S^2$ and prove that $\deg f$ as you present it is well-defined and independent of any choices you need to make in your definition.
(b) Prove in detail that for each integer k (possible negative), there is a C^∞ map $f : S^2 \to S^2$ of degree k.

Hint: $\int_{S^n} F^*\omega = \int_{F_* S^n} \omega$ since we can view S^n as an n-cycle in the domain S^n. Construct $f_k : S^n \to S^n$ of degree k using $z \mapsto z^k$ in S^1 and suspending.

(a) This is exactly Spring 2013-7.

(b) This is exactly Fall 2022-3.

Spring 2010-9. Explain how Stokes Theorem for manifolds with boundary gives, as a special case, the classical divergence theorem (about $\iiint_U \operatorname{div} V\, d(\text{vol})$, where U is a bounded open set in $\mathbb{R}^3$ with smooth boundary and V is a C^∞ vector field on $\mathbb{R}^3$).

Hint: $\iiint_M \operatorname{div}(X) dV = \iint_{\partial M} \langle X, n\rangle dS$ for n the outward pointing normal unit vector to ∂M and $dS = i^* \iota_n dV$ the induced volume form on ∂M. Recall $\operatorname{div}(X) dV = L_X(dV)$ and use Cartan's magic formula. Consider $Y = X - \langle X, n\rangle n$.

This is exactly Fall 2019-1.

Spring 2010-10. (a) Show that every map $F : S^n \to S^1 \times \cdots \times S^1$ (k copies of S^1) is null-homotopic (homotopic to a constant map).

(b) Show that there is a map $F : S^1 \times \cdots \times S^1$ (n copies) $\to S^n$ such that F is not null-homotopic.

(c) Show that every map $F : S^n \to S^{n_1} \times S^{n_2} \times \cdots \times S^{n_k}, n_1 + \cdots + n_k = n, n_j > 0, k \geqslant 2$, has degree 0. (You may use any definition of degree you like, and you may assume F is C^∞.)

Hint: $\pi_1(S^n) = 0$ (assuming $n > 1$) so F lifts to $\widetilde{F} : S^n \to \mathbb{R}^k$, straight-line homotopy shows F is nullhomotopic. One point compactification, map an open neighborhood diffeomorphically to $S^n - \{N\}$. (c) Show that F^* sends a volume form to an exact form, implying that it must be zero.

Referenced in: Fall 2008-9, Spring 2008-6.

(a) We must assume $n > 1$ or else this is false. This is then an easy generalization of Spring 2018-8.

(b) Write T^n for $S^1 \times \cdots \times S^1$. Let $U \subset T^n$ be some open set diffeomorphic to $\mathbb{R}^n$ and let $V = S^n - \{N\}$ be the n-sphere without the north pole, noting that $V \cong \mathbb{R}^n$ as S^n is the one-point compactification of $\mathbb{R}^n$. Define $F : T^n \to S^n$ by letting F map U diffeomorphically onto V and sending all of $T^n - U$ to N. It is easy to see that this is continuous by construction. Moreover, we can make it smooth if necessary by the Whitney approximation theorem.

Then, only points in V can be regular values of F and any such point is necessarily contained in an open neighborhood onto which F is a diffeomorphism. In particular, $F^{-1}(x)$ contains only one point for any $x \in V$, showing that $\deg(F) = 1$. But we know that the degree of a map is homotopy invariant and a constant map has degree 0 so F cannot be nullhomotopic.

(c) Let π_i denote the projection $S^{n_1} \times \cdots \times S^{n_k} \to S^{n_i}$ and let ω_i be a non-vanishing closed n_i-form on S^{n_i}. Then, take

$$\omega = \pi_1^* \omega_1 \wedge \cdots \wedge \pi_k^* \omega_k,$$

so that ω is a non-vanishing closed n-form on $S^{n_1} \times \cdots \times S^{n_k}$. Now, we have

$$F^* \omega = F^* \pi_1^* \omega_i \wedge \cdots \wedge F^* \pi_k^* \omega_k.$$

We know that $n_i < n$ for all i so $H^{n_i}_{dR}(S^n) = 0$, implying that every closed n_i-form on S^n, such as $F^* \pi_i^* \omega_i$, is exact. Let θ_i be such that $F^* \pi_i^* \omega_i = d\theta_i$ for all i so

$$F^* \omega = d\theta_1 \wedge \cdots \wedge d\theta_k = d(\theta_1 \wedge d\theta_2 \wedge \cdots \wedge d\theta_k)$$

implying that $F^* \omega$ is exact. Then by Stokes' theorem, since $\partial S^n = 0$, we have $\int_{S^n} F^* \omega = 0$. In particular, if we let the ω_i be volume forms on S^{n_i} respectively, then ω is a volume form with $\int_{S^2} \omega \neq 0$, implying that $\deg(F) = 0$ by the definition of F^* acting as multiplication by $\deg(F)$ on top cohomology.

Fall 2009

Apparently, this exam doesn't exist!

Spring 2009

Spring 2009-1. (a) Show that a closed 1-form θ on $S^1 \times (-1, 1)$ is dF for some function $F : S^1 \times (-1, 1) \to \mathbb{R}$ if and only if $\int_{S^1} i^*\theta = 0$ where $i : S^1 \to S^1 \times (-1, 1)$ is defined by $i(p) = (p, 0)$ for $p \in S^1$.

(b) Show that a 2-form ω on S^2 is $d\theta$ for some 1-form θ on S^1 if and only if $\int_{S^2} \omega = 0$.

Hint: Define $g(x) = \int_{\gamma_x} \omega$ where γ_x is path from x_0 to x. Well-defined since integrating any loop gives 0 by assumption. Then $dg = \omega$. (b) Split into $A = S^2 - \{N\}$ and $B = S^2 - \{S\}$ which are homotopy equivalent to $\mathbb{R}^2$ so we can find an $\omega_{A,B}$ on each of these. Glue together using the S^1 version of this result which says since $\omega_A - \omega_B$ is closed, it is exact on the intersection so we can make ω global via a bump function.

Referenced in: Spring 2008-4.

(a) Note that $S^1 \times (-1, 1)$ is homotopic to S^1 so $i^* : H^k(S^1 \times (-1, 1)) \to H^k(S^1)$ is an isomorphism for all k, where i^* is induced by the inclusion $i : S^1 \hookrightarrow S^1 \times (-1, 1)$. Thus, a closed form ω on $S^1 \times (-1, 1)$ is exact if and only if $i^*\omega$ is exact on S^1. But then, this reduces to Spring 2019-5.

(b) This is exactly Spring 2011-4. The other direction is a simple application of Stokes' theorem.

Spring 2009-2. Suppose that M, N are connected C^∞ manifolds of the same dimension $n \geqslant 1$ and $F : M \to N$ is a C^∞ map such that $dF : T_pM \to T_{F(p)}N$ is surjective for each $p \in M$.

(a) Prove that if M is compact, then F is onto and F is a covering map.

(b) Find an example of such an everywhere nonsingular equidimensional map where N is compact, F is onto, $F^{-1}(p)$ is finite for each $p \in N$, but F is not a covering map. [A clearly explained pictorial version of F will be acceptable; you do not need to have a "formula" for F.]

Hint: Submersion is open map so onto. Local diffeomorphism implies locally even covering and then piece together using compact/connected. $e^{2\pi i t} : (-0.1, 1.1) \to S^1$.

This is exactly Spring 2018-1.

Spring 2009-3. (a) Suppose that M is a C^∞ connected manifold. Prove that, given an open subset U of M and a finite set of points $p_1, p_2, \ldots, p_k$ in M, there is a diffeomorphism $F : M \to M$ such that $f(\{p_1, p_2, \ldots, p_k\}) \subset U$. (Suggestion: Construct F one point at a time.)

(b) Use part (a) to show that if M is compact and the Euler characteristic $\chi(M) = 0$, then there is a vector field on M which vanishes nowhere. You may assume that if a vector field has isolated zeros, then the sum of the indices at the zero points equals $\chi(M)$.

Hint: Do each $x_i \mapsto y_i$ separately and compose. Connectedness is required. Sum of indices of isolated zeros is zero. Put them all in a neighborhood diffeomorphic to a ball and the function on the boundary $X_p/||X_p||$ has degree zero so can be extended into the ball. Define new vector field based on this.

(a) This is a simpler version of Fall 2020-1.

(b) This is exactly part (a) of Fall 2017-7.

Spring 2009-4. A smooth vector field V on $\mathbb{R}^3$ is said to be "gradient-like" if, for each $p \in \mathbb{R}^3$, there is a neighborhood U_p of p and a function $\lambda_p : U_p \to \mathbb{R} - \{0\}$ such that $\lambda_p V$ on U_p is the gradient of some C^∞ function on U_p. Suppose V is nowhere zero on $\mathbb{R}^3$. Then show that V is gradient-like if and only if $\mathrm{curl}(V)$ is perpendicular to V at each point of $\mathbb{R}^3$.

Hint: $V = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}$ is gradient-like if and only if $\omega \wedge d\omega = 0$ for $\omega = Pdx + Qdy + Rdz$ using the fact that $\ker(\omega)$ is integrable and we can take $V = f\frac{\partial}{\partial z}$. Then just compute $\omega \wedge d\omega$ and $V \cdot \operatorname{curl}(V)$.

By part (b) Spring 2012-4 of, we know that $V = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}$ is gradient-like if and only if $\omega \wedge d\omega = 0$ for $\omega = Pdx + Qdy + Rdz$. Then, note that

$$\operatorname{curl}(V) = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\frac{\partial}{\partial x} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\frac{\partial}{\partial y} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\frac{\partial}{\partial z},$$

so

$$V \cdot \operatorname{curl}(V) = P\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + Q\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + R\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right).$$

On the other hand,

$$\begin{aligned}
\omega \wedge d\omega &= (Pdx + Qdy + Rdz) \wedge (dP \wedge dx + dQ \wedge dy + dR \wedge dz) \\
&= (Pdx + Qdy + Rdz) \\
&\wedge \left(\frac{\partial P}{\partial y}dy \wedge dx + \frac{\partial P}{\partial z}dz \wedge dx + \frac{\partial Q}{\partial x}dx \wedge dy + \frac{\partial Q}{\partial z}dz \wedge dy + \frac{\partial R}{\partial x}dx \wedge dz + \frac{\partial R}{\partial y}dy \wedge dz\right) \\
&= \left(-R\frac{\partial P}{\partial y} + Q\frac{\partial P}{\partial z} + R\frac{\partial Q}{\partial x} - P\frac{\partial Q}{\partial z} - Q\frac{\partial R}{\partial x} + P\frac{\partial R}{\partial y}\right) dx \wedge dy \wedge dz \\
&= \left(\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + Q\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + R\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\right) dV
\end{aligned}$$

so indeed $\omega \wedge d\omega = 0$ if and only if $V \cdot \operatorname{curl}(V) = 0$, i.e., if and only if $\operatorname{curl}(V)$ and V are perpendicular.

Spring 2009-5. Suppose that M is a compact C^∞ manifold of dimension n.

(a) Show that there is a positive integer k such that there is an immersion $F : M \to \mathbb{R}^k$.
(b) Show that if $k > 2n$, there is a $(k-1)$-dimensional subspace H of $\mathbb{R}^k$ such that $P \circ F$ is an immersion, where $P : \mathbb{R}^k \to H$ is orthogonal projection.

Hint: Finite cover, collection of bump functions.

$F : M \to \mathbb{R}^{N(m+1)}, x \mapsto (\lambda_1(x)\phi_1(x), \ldots, \lambda_k(x)\phi_k(x), \lambda_1(x), \ldots, \lambda_k(x))$. Orthogonal complement of $\operatorname{Span}(a)$ for a a regular value not in the image of $g : TM \to \mathbb{R}^L, (p, v) \mapsto DF_p(v)$.

(a) This is exactly Fall 2011-1.

(b) This is done in Fall 2016-2.

Spring 2009-6. Let $Gl^+(n, \mathbb{R})$ be the set of $n \times n$ matrices with determinant > 0. Note that $Gl^+(n, \mathbb{R})$ can be considered to be a subset of $\mathbb{R}^{n^2}$ and this subset is open.

(a) Prove that $Sl^+(n, \mathbb{R}) = \{A \in Gl^+(n, \mathbb{R}) \mid \det(A) = 1\}$ is a submanifold.
(b) Identify the tangent space of $Sl^+(n, \mathbb{R})$ at the identity matrix I_n.
(c) Prove that, for every $n \times n$ matrix B, the series $I_n + B + \frac{1}{2}B^2 + \frac{1}{3!}B^3 + \cdots + \frac{1}{n!}B^n + \cdots$ converges to some $n \times n$ matrix. Notation: this sum $= e^B$.
(d) Prove that if $e^{tB} \in Sl^+(n, \mathbb{R})$ for all $t \in \mathbb{R}$, then $\operatorname{Tr}(B) = 0$.
(e) Prove that if $\operatorname{Tr}(B) = 0$, then $e^B \in Sl^+(n, \mathbb{R})$. [Suggestion: Use one-parameter subgroups or note that it suffices to treat complex-diagonalizable B since such are dense.]

Hint: Preimage theorem with $A \mapsto \det(A)$. $T_I(SL(n, \mathbb{R}) = \ker(d(\det)_I) = \{B \in M_n(\mathbb{R}) \mid \operatorname{Tr}(B) = 0\}$. Operator norm satisfies $||A^n|| \leqslant ||A||^n$. Show $\det(\exp(A)) = \exp(\operatorname{Tr}(A))$ for (d) and (e).

Referenced in: Spring 2008-5.

(a) and (b) This is exactly Fall 2010-3 (having $Gl^+(n, \mathbb{R})$ instead of $M_n(\mathbb{R})$ changes nothing).

(c) Let $||\cdot||$ be the operator norm for matrices, i.e., $||A|| = \sup\{|Ax| \mid |x| = 1\}$. We know this norm satisfies $||A^n|| \leqslant ||A||^n$ so we have

$$0 \leqslant \left|\left|\frac{A^n}{n!}\right|\right| \leqslant \frac{||A||^n}{n!} \implies 0 \leqslant ||\exp(A)|| \leqslant \exp(||A||),$$

where we have also used the triangle inequality. Now, $||A||$ is just a finite number, so $\exp(||A||) < \infty$, showing that the sequence converges to some $n \times n$ matrix since the space of real matrices is complete.

(d) For this, we claim that

$$\det(\exp(A)) = \exp(\mathrm{Tr}(A)).$$

This is true for a diagonal matrix $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ since $\exp(D)$ is diagonal with diagonal entries $(\exp(\lambda_1), \ldots, \exp(\lambda_2))$ so

$$\det(\exp(D)) = \prod_{i=1}^{n} \exp(\lambda_i) = \exp\left(\sum_{i=1}^{n} \lambda_i\right) = \exp(\mathrm{Tr}(D)).$$

Then, for a diagonalizable matrix $A = PDP^{-1}$, we note that $A^n = (PDP^{-1})^n = PD^pP^{-1}$ implying that $\exp(A) = P\exp(D)P^{-1}$ so

$$\det(\exp(A)) = \det(P\exp(D)P^{-1}) = \det(\exp(D)) = \exp(\mathrm{Tr}(D)) = \exp(\mathrm{Tr}(A)),$$

where we have used the multiplicity of det, the previous result, and the fact that trace is independent of base change. Then, since diagonalizable matrices are dense in all matrices, we have shown the claim for all matrices. Now, $e^{tB} \in Sl^+(n, \mathbb{R})$ implies that $\det(e^{tB}) = 1$ so $\exp(\mathrm{Tr}(tB)) = 1$ for all t by the above claim. In particular, we have $\exp(t\mathrm{Tr}(B)) = 1$ since trace is linear which implies that $\mathrm{Tr}(B) = 0$.

(e) This follows immediately from the claim in part (d).

Spring 2009-7. (a) Define complex projective space $\mathbb{CP}^n$.
(b) Calculate the homology of $\mathbb{CP}^n$. Any systematic method such as Mayer-Vietoris or cellular homology is acceptable.

Hint: Attach by $\phi_n : e^{2n} \to \mathbb{CP}^n$, $(z_0, \ldots, z_{n-1}) \mapsto [z_0, \ldots, z_{n-1}, t]$, $t = \sqrt{1 - \sum_{i=1}^{n-1} z_i\overline{z_i}}$. $H_*(\mathbb{CP}^n) = \mathbb{Z}_{(0)} \oplus \mathbb{Z}_{(2)} \oplus \cdots \oplus \mathbb{Z}_{(2n)}$.

This is exactly Spring 2021-5.

Spring 2009-8. Let $p : E \to B$ be a covering space and $f : X \to B$ a map. Define $E^* = \{(x, e) \in X \times B \mid f(x) = p(e)\}$. Prove that $q : E^* \to X$ defined by $q(x, e) = x$ is a covering space.

Hint: Evenly cover $f(y)$. Define $W_\alpha = q^{-1}(U_\alpha)$ where $q((y, \widetilde{x})) = \widetilde{x}$. Then, $z \mapsto (z, p|_{U_\alpha}^{-1} f(z))$ is inverse to $\pi|_{W_\alpha}$.

This is exactly part (a) Spring 2017-6 (note that we didn't use any of the assumptions in part (a)).

Spring 2009-9. (a) Explain carefully and concretely what it means for two (smooth) maps of S^1 into $\mathbb{R}^2$ to be transversal.
(b) Do the same for maps of S^1 into $\mathbb{R}^3$.
(c) Explain what it means for transversal maps to be "generic" and prove that they are indeed generic in the cases of (a) and (b).

Hint: The two loops must not intersect tangentially. No intersection. We can perturb the maps by an arbitrarily small amount so as to make them transversal. In our cases, this boils down to moving them apart or bit or making the intersection non-tangential if it is.

(a) In general, we say that two submanifolds $X, Y \subset Z$ are transversal if they satisfy

$$T_xX + T_xY = T_xZ, \text{ for all } x \in X \cap Y.$$

For two maps $f, g : S^1 \to \mathbb{R}^2$ to be transversal, we require that for all $x \in \text{im}(f) \cap \text{im}(g)$ and any $y \in f^{-1}(x), z \in g^{-1}(x)$, we have

$$df_y(T_yS^1) + dg_z(T_zS^1) = T_x\mathbb{R}^2.$$

In this specific case, we note that $\dim(T_xS^1) = \dim(T_zS^1) = 1$ while $\dim(T_x\mathbb{R}^2) = 2$ so this condition is equivalent to saying that for any $v \in T_yS^1, w \in T_zS^1$ neither of $df_y(v)$ or $dg_z(w)$ is a scalar multiple of the other (so that the pair is linearly independent and spans a two dimensional vector space). In other words, this means that the two loops defined by $f(S^1)$ and $g(S^1)$ must not intersect tangentially.

(b) In this case, note that we can never have $df_y(T_yS^1) + dg_z(T_zS^1) = T_x\mathbb{R}^3$ since $3 > 1 + 1$ so instead, we must have this being vacuously true. I.e., there is no pair $x, y \in S^1$ such that $f(x) = g(y)$ which occurs if and only if the two loops do not intersect at all.

(c) Generic means that, given any two maps, they may be perturbed by an arbitrarily small amount so that they are transverse. In fact, we know that whenever any smooth function is not transverse to a smooth submanifold, we can perturb it an arbitrarily small amount so that it is transverse but we will only prove it in the two cases above.

For case (a), suppose first that f and g intersect tangentially on a discrete set. Then, we may take small neighborhoods around each point that are pairwise disjoint. In these neighborhoods, we may take one loop and move it past the other, creating two non-tangential intersections which are thus transverse. If they intersect tangentially on a non-discrete set, then these problem areas are open sets on which the two loops overlap. Here, we can simply translate one map slightly to pull these areas apart so they don't intersect and are trivially transverse at these points.

For case (b), we do the same thing, using the extra dimension. If f and g intersect non-tangentially at $f(x) = g(y) = z \in \mathbb{R}^3$, then $df_x(T_xS^1) + dg_y(T_yS^1)$ is a plane in $T_z\mathbb{R}^3$ which we may identify with a neighborhood of z. Then, we may pull the loops apart in the direction perpendicular to this plane so that they are no longer intersecting and are thus transverse, completing the proof.

Spring 2009-10. Let M be the 3-manifold with boundary obtained as the union of the two-holed torus in 3-space and the bounded component of its complement. Let X be the space obtained from M by deleting k points from the interior of M.

(a) Calculate the fundamental group of X.
(b) Calculate the homology of X.

Hint: Homotopy equivalent to $S^1 \vee S^1 \vee \bigvee_{i=1}^k S^2$. So $\pi_1(X) = \mathbb{Z} * \mathbb{Z}$ and $H_*(X) = \mathbb{Z}_{(2)}^k \oplus \mathbb{Z}_{(1)}^2 \oplus \mathbb{Z}_{(0)}$.

(a) With a little bit of mental gymnastics, one can visualize a homotopy equivalence between X and $S^1 \vee S^1 \vee \bigvee_{i=1}^k S^2$ as follows. If we were removing 0 points, we can just flatten X down onto the wedge sum of two circles. For each point that we remove, we have to avoid a closed ball surrounding that point when we do our flattening. In so doing, we would be left with the wedge sum of two circles onto which are glued k unit 3-balls with holes in them that we can see each deformation retract onto S^2. Thus, we can immediately calculate using Van Kampen's theorem:

$$\pi_1(X) = \pi_1(S^1 \vee S^1 \vee \bigvee_{i=1}^k S^2) = \pi_1(S^1) * \pi_1(S^1) * *_{i=1}^k \pi_1(S^2) = \mathbb{Z} * \mathbb{Z}.$$

(b) Using the same homotopy equivalence noted in part (a), we can immediately see that

$$H_i(X) = \begin{cases} \mathbb{Z} & i = 0, \\ \mathbb{Z}^2 & i = 1, \\ \mathbb{Z}^k & i = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Spring 2009-11. Let P be a finite polyhedron.

(a) Define the Euler characteristic $\chi(P)$ of P.

(b) Prove that if P_1, P_2 are subpolyhedra of P such that $P_1 \cap P_2$ is a point and $P_1 \cup P_2 = P$, then $\chi(P) = \chi(P_1) + \chi(P_2) - 1$.

(c) Suppose that $p : E \to P$ is an n-sheeted covering space of P, that is $p^{-1}(x)$ is n points for each $x \in P$. Prove that $\chi(E) = n\chi(P)$.

Hint: Alternating sum of number of i-simplices. Inclusion-exclusion. Lift characteristic maps, n k-cells for each k-cell in X.

(a) The Euler characteristic is

$$\chi(P) = \sum_{i=0}^{n} (-1)^i \text{Rank}(H_i(P)) = \sum_{i=0}^{n} (-1)^i (\#i\text{-simplices}).$$

(b) This is a simple inclusion-exclusion. $\chi(P_1) + \chi(P_2)$ will count over all simplices in P since $P_1 \cup P_2 = P$, but will double count the single point in $P_1 \cap P_2$ accounting for the -1 in the formula for $\chi(P)$.

(c) This is exactly part (a) of Spring 2023-6.

Spring 2009-12. Let $f : T \to T = S^1 \times S^1$ be a map of the torus inducing $f_\pi : \pi_1(T) \to \pi_1(T) = \mathbb{Z} \oplus \mathbb{Z}$ and let F be a matrix representing f_π. Prove that the determinant of F equals the degree of the map f.

Hint: Take abelianization to get map on homology, dualize to get map on cohomology with transpose matrix representing it. Use Künneth theorem and cup product structure on $H^*(T)$ to deduce the result

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Taking abelianization, we note that the induced homomorphism on first homology

$$f_* : H_1(T) \to H_1(T) = \mathbb{Z} \oplus \mathbb{Z}$$

is represented by precisely the same matrix F since $\pi_1(T) = H_1(T)$ is already abelian. Since the ith homology of T is free and finitely generated for all i, we can use the univerisal coefficient theorem to dualize in all dimensions, obtaining a map

$$f^* : H^*(T) \to H^*(T).$$

In particular, $f^* : H^1(T) \to H^1(T)$ is represented by the transpose matrix F^* of F. Let x, y be a basis for $H^1(T)$ with respect to which, F^* is given by

$$F^* = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then, by Künneth's theorem the cup product structure on $H^*(T)$ can be described as follows. In $H^2(T), x^2 = y^2 = 0$ and xy is a generator with $xy = -yx$. Then we may compute (using the above matrix)

$$f^*(xy) = (f^*x)(f^*y) = (ax + cy)(bx + dy) = abx^2 + adxy + bcyx + cdy^2 = (ad - bc)xy.$$

One definition of degree is that $f^* : H^2(T) = \mathbb{Z} \to H^2(T) = \mathbb{Z}$ is just multiplication by $\deg(f)$. So since xy is a generator for $H^2(T)$, we have $\deg(f) = ad - bc = \det F^* = \det F$ as desired.

Fall 2008

Fall 2008-1. Let $G(k,n)$ be the collection of all k-dimensional linear subspaces in $\mathbb{R}^n$.

(a) Define a natural topological and smooth structure on $G(k,n)$, and show that with respect to the structures you defined, $G(k,n)$ is a compact smooth manifold.
(b) Show that $G(k,n)$ is diffeomorphic to $G(n-k,n)$.

Hint: $Gr(k,n) \sim \text{Mat}^k_{n\times k}(\mathbb{R})/\sim$. Define charts on the subsets with each certain $k\times k$ submatrix invertible by reading out the other rows of $AA^{-1}_{i_1,\dots,i_k}$. Dimension is $(n-k)\times k$. Dimension is $(n-k)\times k$. Compact as it is the quotient of the Stiefel manifold $V_k(\mathbb{R}^n)$. (b) Send space to its orthogonal complement.

(a) This is exactly Fall 2022-1. To see that $G(k,n)$ is compact, note that it is a quotient of the Stiefel manifold $V_k(\mathbb{R}^n)$ of orthonormal vectors $v_1,\dots,v_k\in\mathbb{R}^n$ given by identifying two lists of vectors if they span the same k-dimensional subspace of $\mathbb{R}^n$. Then, $V_k(\mathbb{R}^n)$ is compact as it is a closed and bounded subset of $\mathbb{R}^{nk}$.

(b) The diffeomorphism is given by sending a space to its orthogonal complement. Bijectivity is clear and smoothness follows from the definition of the charts in part (a) though we will not check all the details.

Fall 2008-2. Let M and N be two smooth manifolds, and $f: M\to N$ be a smooth map. Assume that $df_x: T_xM\to T_{f(x)}N$ is surjective for all x in M and that the inverse image $f^{-1}(y)$ is compact for all y in N.

(a) Show that for any y in N there is an open neighborhood V of y such that $f^{-1}(V)$ is diffeomorphic to $V\times f^{-1}(y)$.
(b) Assume further that N is connected, can you take V to be N in (a)? (Justify your answer.)

Hint: (a) is actually false, say by $\exp: (0,1)\hookrightarrow S^1$. Even if we were to assume that f is proper (so that part (a) is true and f is a covering map), we have the counterexample $S^1\to S^1\subset\mathbb{C}$ given by $z\mapsto z^2$.

(a) This is false. Consider $f:(0,1)\to S^1\subset\mathbb{C}$ given by $x\mapsto e^{2\pi i x}$. Then, $f^{-1}(1)=\varnothing$ and $f^{-1}(z)$ is a single point for any other $z\in S^1-\{1\}$ so $f^{-1}(z)$ is indeed compact. Moreover, f is clearly smooth and a local diffeomorphism so df_x is indeed a surjection for all $x\in(0,1)$. But any neighborhood V of $1\in S^1$ contains a point in the image of f so $f^{-1}(V)\neq\varnothing$ while $f^{-1}(1)=\varnothing$ so $V\times f^{-1}(1)=\varnothing$ which is not correct.

(b) No we cannot. Consider the two-fold cover $f: S^1\to S^1$ given by $z\mapsto z^2$. This satisfies all of the conditions and, in fact, f is a covering map, but we still cannot take V to be N. Clearly $f^{-1}(S^1)=S^1\neq S^1\times f^{-1}(y)=S^1\times\{\sqrt{y},-\sqrt{y}\}$.

Fall 2008-3. Let M be a connected smooth manifold. Show that for any two points x and y in M there is a diffeomorphism f of M such that $f(x)=y$.

Hint: Do it in $\mathbb{R}^n$ first and compose along a path.

This is an easier version of Fall 2020-1.

Fall 2008-4. Let $\theta=\sum_{i=1}^n(x_idy_i-y_idx_i)$ be a 1-form defined on $\mathbb{R}^{2n}$, where $(x_1,\dots,x_n,y_1,\dots,y_n)$ are the coordinates of $\mathbb{R}^{2n}$. Consider the $2n-1$ dimensional distribution $D=\ker\theta$. Is D integrable? (Justify your answer.)

Hint: Use if and only if $\theta\wedge d\theta=0$ condition. Not 0 so not integrable.

By Fall 2022-4, we simply check if $\theta \wedge d\theta = 0$. Note that

$$d\theta = \sum_{i=1}^{n} dx_i \wedge dy_i - dy_i \wedge dx_i = 2\sum_{i=1}^{n} dx_i \wedge dy_i.$$

Then,

$$\begin{aligned}
\theta \wedge d\theta &= 2\sum_{i,j=1}^{n} x_i dy_i \wedge dx_j \wedge dy_j - y_i dx_i \wedge dx_j \wedge dy_j \\
&= 2\sum_{i \neq j} x_i dy_i \wedge dx_j \wedge dy_j - y_i dx_i \wedge dx_j \wedge dy_j \\
&= 2\sum_{i<j} -x_i dx_j \wedge dy_i \wedge dy_j - y_i dx_i \wedge dx_j \wedge dy_j + 2\sum_{i>j} x_i dx_j \wedge dy_j \wedge dy_i + y_i dx_i \wedge dx_i \wedge dy_j \\
&\neq 0
\end{aligned}$$

so $\ker(\theta)$ is not integrable.

Fall 2008-5. Let D be a bounded domain in $\mathbb{R}^n$ with a smooth boundary S, $j : S \to \mathbb{R}^n$ be the inclusion map and X be a smooth vector field defined on $\mathbb{R}^n$.

(a) Denote the standard volume form $dx_1 \wedge \cdots \wedge dx_n$ by ω. Show that $j^*(i_X\omega) = \langle X, N\rangle dS$, where N is the outer unit normal vector field along S, $\langle X, N\rangle$ is the Euclidean inner product of X and N. Here $i_X\omega$ is the contraction of ω along X, dS is the "area" form on S. Explain carefully the definition and geometrical meaning of the term "dS".

(b) Use (a) and Stokes theorem to show that

$$\int_D L_X\omega = \int_S \langle X, N\rangle dS.$$

Here $L_X\omega$ is the Lie derivative of ω along X.

Hint: $dS = \iota^*(i_N(\omega))$. (b) $T = X - \langle X, N\rangle N$ is tangent to M so any $T, d\iota X_1, \ldots, d\iota X_{n-1}$ is linearly dependent so $\iota^*(i_T(dV))(X_1, \ldots, X_{n-1}) = 0$. Stokes' and Cartan.

This is exactly parts (a), (b), and (c) of Fall 2018-5.

Fall 2008-6. Find $\pi_1(T^2 - \{k \text{ pts}\})$, where T^2 is the two-dimensional torus.

Hint: Wedge sum of $k+1$ circles.

Note that if $k = 1$, then $T^2 - \{p\}$ deformation retracts onto the wedge of two circles. If we remove more points, then it deformation retracts onto the wedge sum of $k+1$ circles so the fundamental group is $*_{i=1}^{k+1}\mathbb{Z}$, the free product of $k+1$ copies of $\mathbb{Z}$.

Fall 2008-7. Find the homology groups $H_i(\Delta_n^{(k)}), i = 0, 1, \ldots, k$. Here $\Delta_n^{(k)}$ is the k-skeleton of the n-simplex Δ_n with $k \leqslant n$.

Hint: $\widetilde{H}_*(\Delta_n^{(k)}) = \mathbb{Z}_{(k)}^{\binom{n}{k+1}}$ by induction using long exact sequence coming from the mapping cone for inclusion $\Delta_{n-1}^{(k-1)} \hookrightarrow \Delta_{n-1}^{(k)}$. Unreduced has one extra copy of $\mathbb{Z}$ in degree 0.

This is exactly Spring 2021-4 (except there we did reduced homology which makes the formula look a bit cleaner).

Fall 2008-8. Let G be a topological group with the identity element e. For any two continuous loops γ_1 and $\gamma_2 : S^2 \to G$ sending $1 \in S^1$ to $e \in G$, define $\gamma_1 * \gamma_2 : S^1 \to G$ by $\gamma_1 * \gamma_2(t) = \gamma_1(t) \circ \gamma_2(t)$ for $t \in S^1$. Here $\circ$ is the product operation in G.

(a) Show that the product $*$ so defined induces a product structure on $\pi_1(G, e)$ and that this new product on $\pi_1(G, e)$ is the same as the usual one.
(b) Is $\pi_1(G, e)$ commutative? (Justify your answer.)

Hint: Reparameterize with one of the two loops being the identity for the first half, and the second being the identity for the second half.

This is exactly Fall 2010-8.

Fall 2008-9. (a) Show that any continuous map $f : S^2 \to T^2$ is null-homotopic.
(b) Show that there exists a continuous map $f : T^2 \to S^2$ which is not null-homotopic.

Hint: $\pi_1(S^2) = 0$ so f lifts to $\widetilde{F} : S^2 \to \mathbb{R}^2$, straight-line homotopy shows F is nullhomotopic. One point compactification, map an open neighborhood diffeomorphically to $S^2 - \{N\}$.

This is a specific case of parts (a) and (b) of Spring 2010-10.

Fall 2008-10. Let A and B be two chain complexes with boundary operators ∂_A and ∂_B respectively, and $f : A \to B$ be a chain map. Define a new chain complex C whose ith chain group is $C_i = A_i \bigoplus B_{i+1}$ and whose boundary operator ∂_C is defined by $\partial_C(a, b) = (\partial_A(a), \partial_B(b) + (-1)^{\deg(a)} f(a))$ for any $(a, b) \in C_i$. Here A_i and B_i are the ith chain groups of A and B respectively.

(a) Show that C so defined is indeed a chain complex and that there is a short exact sequence of chain complexes:
$$0 \to B \to C \to A \to 0$$
sending B_{i+1} to C_i and C_i to A_i.
(b) Write down the long exact sequence of the homology groups associated to the short exact sequence in (a). What is the connecting boundary map in the long exact sequence?
(c) Let $(f_*)_i : H_i(A) \to H_i(B)$ be the induced map of f on the ith homology groups. Show that $(f_*)_i : H_i(A) \to H_i(B)$ is an isomorphism for all i if and only if $H_i(C) = 0$ for all i.

Hint: Check $\partial_C^2 = 0$, $g_i : B_{i+1} \to C_i$ and $h_i : C_i \to A_i$ are chain maps (commute with differential), and this is an exact sequence. Snake lemma: $\delta_i(a) = (-1)^i f_i(a)$. Look at long exact sequence.

(a) First, we show $\partial_C^2 = 0$.

$$\begin{aligned}
\partial_C^2(a, b) &= \partial_C(\partial_A(a), \partial_B(b) + (-1)^{\deg(a)} f(a)) \\
&= (\partial_A^2(a), \partial_B(\partial_B(b) + (-1)^{\deg(a)} f(a)) + (-1)^{\deg(\partial_A(a))} f(\partial_A(a))) \\
&= (0, \partial_B^2(b) + (-1)^{\deg(a)} \partial_B(f(a)) + (-1)^{\deg(a)-1} f(\partial_A(a))) \\
&= (0, 0) + (-1)^{\deg(a)}(f(\partial_A(a)) - f(\partial_A(a))) \\
&= (0, 0),
\end{aligned}$$

where we use the facts that $\partial_A^2 = \partial_B^2 = 0$ and that f is a chain map so $\partial_B \circ f = f \circ \partial_A$. Next, we show that $g_i : B_{i+1} \to C_i$ and $h_i : C_i \to A_i$ are themselves chain maps.

$$\begin{aligned}
\partial_C(g_i(b)) &= \partial_C(0, b) = (\partial_A(0), \partial_B(b) + (-1)^i f_i(0)) \\
&= (0, \partial_B(b) + 0) = g_{i-1}(\partial_B(b)) \\
h_i(\partial_C(a, b)) &= h_i(\partial_A(a), \partial_B(b) + (-1)^i f_i(a)) \\
&= \partial_A(a) = \partial_A(h_i(a, b)),
\end{aligned}$$

showing $\partial_C \circ g = g \circ \partial_B$ and $\partial_A \circ h = h \circ \partial_C$ as desired. Finally, we show that this sequence is exact at each i. For this, note that $h_i(g_i(b)) = h_i(0, b) = 0$ so $\text{im}(g_i) \subset \ker(h_i)$. Conversely, let $(a, b) \in \ker(h_i)$, then $a = 0$ and $(a, b) = (0, b) = g_i(b)$ so $\ker(h_i) \subset \text{im}(g_i)$ and we have exactness. Thus, we have a short exact sequence of chain complexes.

(b) The induced sequence is

$$\cdots \to H_{i+1}(B) \to H_i(C) \to H_i(A) \xrightarrow{\delta_i} H_i(B) \to \cdots,$$

where $\delta_i : \ker(\delta_A) \to \text{coker}(\partial_B)$ is the connecting map coming from the snake lemma. In this particular case, we can describe it more explicitly by considering the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 & & & & & & \ker(\partial_A) & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & B_{i+1} & \xrightarrow{g_i} & C_i & \xrightarrow{h_i} & A_i & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_B} & & \downarrow{\scriptstyle \partial_C} & & \downarrow{\scriptstyle \partial_A} & & \\
0 & \longrightarrow & B_i & \xrightarrow{g_{i-1}} & C_{i-1} & \xrightarrow{h_{i-1}} & A_{i-1} & \longrightarrow & 0 \\
 & & \downarrow & & & & & & \\
 & & \text{coker}(\partial_B) & & & & & &
\end{array}$$

Given $a \in \ker(\partial_A)$, we can find $(a', b) \in C_i$ so that $h_i(a', b) = a' = a$ so $a' = a$. By the standard argument of the snake lemma, our output is independent of this choice so we may as well take $b = 0$ for simplicity, i.e., we have $(a, 0) \in C_i$. This gets mapped to $\partial_C(a, 0) = (\partial_A(a), (-1)^i f_i(a)) \in C_{i-1}$ which we know by commutativity of the right square gets sent to 0 by h_{i-1}, namely $\partial_A(a) = 0$ as we already knew.

Thus, by exactness of the bottom row, we know there exists a unique $b' \in B_i$ such that $g_{i-1}(b') = (\partial_A(a), (-1)^i f_i(a))$. We clearly have $b' = (-1)^i f_i(a)$ and so our map is defined by $\delta_i(a) = (-1)^i f_i(a)$ which descends to a map on homology by the snake lemma.

(c) If $H_i(C) = 0$ for all i, then our long exact sequence becomes

$$\cdots \to 0 \to H_i(A) \xrightarrow{\delta_i} H_i(B) \to 0 \to \cdots,$$

at each degree i. By exactness, we immediately conclude that δ_i is an isomorphism. Thus, if i is even, $\delta_i = f_i$ is an isomorphism while if i is odd, $\delta_i = -f_i$ so f_i is still an isomorphism with $(f_i)^{-1} = -\delta_i^{-1}$.

Conversely, if f_i is an isomorphism for all i, then $\delta_i = (-1)^i f_i$ is also an isomorphism for all i, implying that $\ker(\delta_i) = \text{coker}(\delta_i) = 0$ for all i which can only occurs if $H_i(C) = 0$ for all i by considering the long exact sequence.

Spring 2008

Spring 2008-1. Let M and N be smooth (C^∞) manifolds, not necessarily of the same dimension, and $F : M \to N$ be a smooth map.

(a) Define the map F^* of p-forms on N to p-forms on $M, p = 0, 1, 2, \ldots$.
(b) Prove that, if ω is a p-form on N, then $F^*(d_N\omega) = d_M(F^*\omega)$.

Hint: Precomposition of the coefficient functions by F. Show induction for function and then for $\alpha = dg \wedge \eta$. Linearity/locality finishes the proof.

(a) It suffices to give a definition in local coordinates. Let $x_1, \ldots, x_m$ be coordinates on N so that a basis for the space of p-forms on N is given by $dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ for $1 \leqslant i_1 < \cdots < i_p \leqslant n$. So a general form looks like

$$\omega = \sum_{1 \leqslant i_1 < \cdots < i_p \leqslant n} g_{i_1,\ldots,i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p},$$

where $g_{i_1,\ldots,i_p} \in C^\infty(N)$. Then, we define

$$F^*\omega = \sum_{1 \leqslant i_1 < \cdots < i_p \leqslant n} (g_{i_1,\ldots,i_p} \circ F) d(x_{i_1} \circ F) \wedge \cdots d(x_{i_p} \circ F).$$

(b) This is exactly Fall 2011-5.

Spring 2008-2. Let M be a C^∞ manifold and X a C^∞ vector field on M.

(a) Suppose $X(p) \neq 0$ for some particular $p \in M$. Show, using the flow of X, there is a neighborhood U of p and a coordinate system $(x_1, \ldots, x_n)$ on U with $X = \frac{\partial}{\partial x_1}$ on U.

(b) Use part (a) to prove that if Y is another C^∞ vector field on M with $[X, Y] = 0$ everywhere on M, then $\varphi_s(\psi_t(p)) = \psi_t(\varphi_s(p))$ for all s, t with $|t|$ and $|s|$ sufficiently small, where φ, ψ are the flows of X and Y respectively. [Suggestion: Write Y near p in the coordinate system of part (a).]

Hint: Use local flow $\phi : I \times U \to U$, define $\psi(a_1, \ldots, a_n) = \phi(a_1, (0, a_2, \ldots, a_n))$. Show $d\psi_0 = \text{id}$ so inverse function theorem gives local inverse $(y_1, \ldots, y_n) = \psi^{-1}(x_1, \ldots, x_n)$ and this chart is what we want $\frac{\partial}{\partial y_1} = V$. Don't follow suggestion: instead use $L_X Y = \lim_{h\to 0} \frac{Y_q - (\varphi_h^* Y)_q}{h} = 0$ for all q, let $c(t) = (\varphi_t^* Y)_p$ be a curve and show that $c'(t) = 0$ so $\varphi_t^* Y = Y$ and consider flows corresponding to both.

(a) This is exactly Spring 2011-1.

(b) We note that $L_X Y = [X, Y] = 0$. Thus, for all $q \in M$, we have

$$\lim_{h\to 0} \frac{Y_q - (\varphi_h^* Y)_q}{h} = 0.$$

Let c be a curve in M_p given by $c(t) = (\varphi_t^* Y)_p$. Then,

$$\begin{aligned} c'(t) &= \lim_{h\to 0} \frac{c(t+h) - c(t)}{h} \\ &= \lim_{h\to 0} \frac{(\varphi_{t+h}^* Y)_p - (\varphi_t^* Y)_p}{h} \\ &= \lim_{h\to 0} \frac{\varphi_t^* (\varphi_h^* Y)_{\varphi_{-t}^*(p)} - \varphi_t^* Y_{\varphi_{-t}^*(p)}}{h} \\ &= \varphi_t^* \lim_{h\to 0} \frac{(\varphi_h^* Y)_{\varphi_{-t}^*(p)} - Y_{\varphi_{-t}^*(p)}}{h} \\ &= \varphi_t^* 0 = 0, \end{aligned}$$

where in the last line, we took the point $q = \varphi_{-t}^*(p)$ and used the above equality. Hence, this shows that c is constant, i.e., $c(t) = c(0)$ for $|t|$ small enough. Namely, $(\varphi_t^* Y)_p = (\varphi_0^* Y)_p = Y_p$ so the flows corresponding to $\varphi_t^* Y$ and Y are the same. But we know the flow corresponding to $\varphi_t^* Y$ is $\varphi_t \circ \psi_s \circ \varphi_{-t}$ while that of Y is ψ_s, giving us the desired relation.

Spring 2008-3. Gauss's Divergence Theorem asserts that if U is a bounded open set in $\mathbb{R}^3$ with smooth boundary and if X is a smooth vector field defined in a neighborhood of the closure of U, then $\iiint_U \text{div}(X) d(\text{vol}) = \iint_{\partial U} X \cdot N d(\text{area})$ where N is the exterior unit normal to ∂U. Show how the Divergence Theorem follows from Stokes Theorem for differential forms on manifolds with boundary.

Hint: $\iiint_M \text{div}(X) dV = \iint_{\partial M} \langle X, n\rangle dS$ for n the outward pointing normal unit vector to ∂M and $dS = i^* \iota_n dV$ the induced volume form on ∂M. Recall $\text{div}(X) dV = L_X(dV)$ and use Cartan's magic formula. Consider $Y = X - \langle X, n\rangle n$.

This is exactly Fall 2019-1.

Spring 2008-4. (a) Let θ be a 1-form on S^2 with $d\theta = 0$. Construct a function f on S^2 with $df = \theta$.

(b) Let θ be a 1-form on $S^1 \times (0,1)$ with $d\theta = 0$. Show that there is a function $f : S^1 \times (0,1) \to \mathbb{R}$ with $df = \theta$ if and only if $\int_{S^1 \times \{1/2\}} \theta = 0$.

(c) Use part (b) to show that if ω is a 2-form on S^2 with $\int_{S^2} \omega = 0$, then there is a 1-form θ on S^2 with $d\theta = \omega$. [Suggestion: You may assume the Poincaré Lemma so that $\omega = d\theta_1$ on $S^2 - \{\text{South pole}\}$ and $\omega = d\theta_2$ on $S^2 - \{\text{North pole}\}$. Use Stokes theorem to show $\theta_1 - \theta_2$ satisfies the integral condition of part (b).]

Hint: Define $g(x) = \int_{\gamma_x} \omega$ where γ_x is path from x_0 to x. Well-defined since integrating any loop gives 0 by assumption. Then $dg = \omega$. (b) Split into $A = S^2 - \{N\}$ and $B = S^2 - \{S\}$ which are homotopy equivalent to $\mathbb{R}^2$ so we can find an $\omega_{A,B}$ on each of these. Glue together using the S^1 version of this result which says since $\omega_A - \omega_B$ is closed, it is exact on the intersection so we can make ω global via a bump function.

(a) This can actually be proved in the same way as part (c) since we only use there that $d\omega = 0$.

(b) and (c). This is exactly Spring 2009-1.

Spring 2008-5. Let $SO(3)$ = the set of all 3×3 matrices A with AA^t = identity (orthogonal matrices) and determinant of $A = 1$. Also, for each 3×3 matrix B, let

$$\exp(B) = 1 + B + \left(\frac{B^2}{2!}\right) + \left(\frac{B^3}{3!}\right) + \dots.$$

(a) Prove that the infinite series for $\exp(B)$ converges for each 3×3 matrix B, so that exp is a map from the space of 3×3 matrices to itself.
You may *assume* from here on that this map is smooth and that the series can be differentiated term by term to give the differential of the mapping.

(b) Show that the map exp is injective on some neighborhood of the 0 matrix in the space of all 3×3 matrices. [Suggestion: Inverse function theorem.]

(c) Prove that $\exp(B)$ is in $SO(3)$ if B satisfies $B^t = -B$ (B is "anti-symmetric").

(d) Show that the mapping exp restricted to the vector space of 3×3 anti-symmetric matrices is a surjective (onto) map from some neighborhood of the 0 matrix to a neighborhood of the identity matrix in $SO(3)$. [Suggestion: Note that every element of $SO(3)$ is a rotation around an axis, so check this case.]

(e) Discuss how to combine parts (b), (c), and (d) to give coordinate charts on $SO(3)$ and thus to make $SO(3)$ a differentiable manifold.

Hint: Use matrix norm which satisfies $||A^n|| \leqslant ||A||^n$. Differentiate termwise, $d(\exp) = \exp$. Show $\det(\exp(A)) = \exp(\mathrm{Tr}(A))$ and $\exp(A+B) = \exp(A)\exp(B)$ for A, B commuting. Rotation can be sent to by $\begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}$.

(a) This is exactly part (c) of Spring 2009-6.

(b) Differentiating termwise, we get $d(\exp) = \exp$ so $d(\exp)_0(0) = \exp(0) = I_3 \neq 0$ so by the inverse function theorem, exp is invertible in some neighborhood the zero matrix. In particular, it must be injective.

(c) By part (d) of Spring 2009-6, we have $\det(\exp(A)) = \exp(\mathrm{Tr}(A))$ for all matrices A. Thus, if $B^t = -B$, we note that B must have zeros on the diagonal so $\mathrm{Tr}(B) = 0$ and thus $\det(\exp(B)) = \exp(0) = 1$. Moreover, we note that for commuting matrices A and B, the formula for exp tells us that $\exp(A+B) = \exp(A)\exp(B)$ so in particular,

$$\exp(B)\exp(B)^t = \exp(B)\exp(B^t) = \exp(B + B^t) = \exp(B - B) = \exp(0) = I_3,$$

showing that $\exp(B)$ both has determinant 1 and is orthogonal so must be in $SO(3)$.

(d) Using the hint, we let $A \in SO(3)$ be a rotation about an axis, which up to an orthonormal base change, is given by

$$A = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is not too hard to see that (or noting the similarity with complex numbers) taking

$$B = \begin{pmatrix} 0 & -\theta & 0 \\ \theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

gives $\exp(B) = A$ where B is clearly antisymmetric. Then, it is also clear that exp takes a neighborhood of 0 to a neighborhood of the identity since $\exp(0) = I$ and exp is continuous so we are done.

(e) The antisymmetric matrices have a natural smooth structure inherited from $M_3(\mathbb{R})$, the space of all 3×3 matrices with real entries. Then, (b), (c), and (d) gave us a surjective immersion from the antisymmetric matrices to $SO(3)$ which we can use to define the charts on $SO(3)$ so that it has the induced smooth structure.

Spring 2008-6. Let M and N be two compact, oriented manifolds of the same dimension. And let ω be a nowhere vanishing n-form on N with $\int_N \omega = 1$. Let $F : M \to N$ be a smooth map.

(a) Set $\deg_\omega F = \int_M F^*\omega$. Show that $\deg_\omega F$ is independent of the choice of ω. [You may assume de Rham's Theorem.] We shall call the common value the degree of F.
(b) Show that there is a smooth map from $S^2 \times S^2$ to S^4 of degree 1.
(c) Show that no map from S^4 to $S^2 \times S^2$ has degree 1.

Hint: $\omega_1 - \omega_2$ is exact by de Rham, then use Stokes' as M has no boundary. One point compactification, map an open neighborhood diffeomorphically to $S^n - \{N\}$. Show that F^* sends a volume form to an exact form, implying that it must be zero.

(a) Suppose ω_1 and ω_2 are two nowhere vanishing n-forms on N with $\int_N \omega_1 = \int_N \omega_2 = 1$. Then, by de Rham's theorem, we know that ω_1 and ω_2 are in the same cohomology class so $\omega_1 - \omega_2$ is exact, say with $\omega_1 - \omega_2 = d\eta$ for some $(n-1)$-form η. Then,

$$\deg_{\omega_1} F - \deg_{\omega_2} F = \int_M F^*\omega_1 - \int_M F^*\omega_2 = \int_M F^*(\omega_1 - \omega_2) = \int_M F^*(d\eta) = \int_M dF^*(\eta) = \int_{\partial M} F^*(\eta) = 0,$$

by Stokes' theorem and since $\partial M = \varnothing$, showing that the degree is independent of the choice of ω.

(b) Follow the method in part (b) of Spring 2010-10, noting that $S^2 \times S^2$ is a 4-manifold so has a 4-cell.

(c) This is an easier version of part (c) of Spring 2010-10.

Spring 2008-7. Describe carefully the basic algebraic construction of algebraic topology, namely, how to go from a short exact sequence of chain complexes to a long exact sequence in homology. Give explicitly, in particular, the construction of the "connecting homomorphism", the map where the dimension drops, and prove exactness at its image, that is, prove that the image of the connecting homomorphism = the kernel of the map that follows it. [You need not prove exactness of the long exact sequence elsewhere.]

Hint: This is the snake lemma, for which, the art of the diagram chase cannot be hinted at. Show part (b) as two directions.

This is exactly Spring 2012-5.

Spring 2008-8. (a) Prove that S^n is simply connected if $n > 1$.
(b) Prove that $\pi_1(\mathbb{RP}^n) = \mathbb{Z}_2, n > 1$.
(c) Prove that $\mathbb{RP}^n$ is orientable if n is odd ($n > 1$).

Hint: Sard's theorem shows $S^1 \to S^n$ not surjective so factors through map to $\mathbb{R}^n$. CW structure with $\mathbb{RP}^2 \subset \mathbb{RP}^n$. Top homology is $\mathbb{Z}$ if and only if n is even.

(a) First, we note by Sard's theorem that since $n > 1$, any map $f : S^1 \to S^n$ cannot be surjective. This is because every point in the image of f must be a critical value so the image has measure 0. Thus, f factors through a map $f : S^1 \to S^n - \{p\}$ for some $p \in S^n$. But we know $S^n - \{p\}$ is homotopic to $\mathbb{R}^n$ so we can think of f as a map $f : S^1 \to \mathbb{R}^n$ which means f is nullhomotopic via $f_t = tf$, showing that S^n is simply connected.

(b) We know that $\pi_1(\mathbb{RP}^n)$ only depends on the 1- and 2-skeleta of $\mathbb{RP}^n$ which we know to be exactly $\mathbb{RP}^1 \cong S^1$ and $\mathbb{RP}^2$ since we can give $\mathbb{RP}^n$ a CW structure such that its k-skeleton is $\mathbb{RP}^k$ for all $0 \leqslant k \leqslant n$. Then, we know that $\mathbb{RP}^2$ is formed by attaching a 2-cell to a 1-cell via a degree two map, so it is clear that $\pi_1(\mathbb{RP}^2) = \langle a \mid a^2 \rangle = \mathbb{Z}/2\mathbb{Z}$.

(c) We showed this in Spring 2021-6.

Spring 2008-9. Find by any method the homology groups of $\mathbb{RP}^n$ with integer coefficients.

Hint: One cell in each dimension with double cover for attaching maps. $H_k(\mathbb{RP}^n) = \mathbb{Z}$ if $k = n$ and n is odd, $\mathbb{Z}/2\mathbb{Z}$ if $k \leqslant n$ and k is odd and 0 otherwise.

The result in Fall 2020-7 generalizes trivially (noting the difference between n even and n odd).

Spring 2008-10. (a) Define complex projective space $\mathbb{CP}^n$.
(b) Show that $\mathbb{CP}^n$ is compact.
(c) Show that $\mathbb{CP}^1 = S^2$ (homeomorphic is enough).
(d) Show that $\mathbb{CP}^n$ is simply connected.
(e) Find the homology of $\mathbb{CP}^n$ (integer coefficients). [Any method will do. But cell complex decomposition is the easiest.]

Hint: Attach by $\phi_n : e^{2n} \to \mathbb{CP}^n, \quad (z_0, \ldots, z_{n-1}) \mapsto [z_0, \ldots, z_{n-1}, t], \quad t = \sqrt{1 - \sum_{i=1}^{n-1} z_i \overline{z_i}}$. $H_*(\mathbb{CP}^n) = \mathbb{Z}_{(0)} \oplus \mathbb{Z}_{(2)} \oplus \cdots \oplus \mathbb{Z}_{(2n)}$ and $\pi_1(\mathbb{CP}^n) = 0$. Compactness and homeomorphism are immediate.

(a), (d), and (e) are done in Spring 2021-5.

(b) This is immediate since we have given $\mathbb{CP}^n$ a finite CW structure.

(c) By the cell decomposition, we know that $\mathbb{CP}^1$ is formed by gluing a 2-cell to a 0-cell, which is exactly the same CW structure as S^2 so they are homeomorphic.

The end...